Use Gauss-Jordan elimination in order to solve:

$$
\begin{aligned}
2x_1 - x_2 + x_3 - 3x_4 &= 1 \\
5x_1 + 2x_2 - x_3 + x_4 &= 1 \\
3x_1 - 3x_2 - x_3 + 2x_4 &= 3 \\
10x_1 - 2x_2 - x_3 &= 5
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{bmatrix}
2 & -1 & 1 & -3 & 1 \\
5 & 2 & -1 & 1 & 1 \\
3 & -3 & -1 & 2 & 3 \\
10 & -2 & -1 & 0 & 5
\end{bmatrix}
$$

$$
\sim \begin{array}{r} \\ 2R_2 \\ 2R_3 \\ \\ \end{array}
\begin{bmatrix}
2 & -1 & 1 & -3 & 1 \\
10 & 4 & -2 & 2 & 2 \\
6 & -6 & -2 & 4 & 6 \\
10 & -2 & -1 & 0 & 5
\end{bmatrix}
$$

We mult. row 2&3 by 2 since upon inspection no "easy" elementary row operation is possible using the first row

$$
\sim \begin{array}{r} \\ -5R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \\ -5R_1 + R_4 \to R_4 \end{array}
\begin{bmatrix}
2 & -1 & 1 & -3 & 1 \\
0 & 9 & -7 & 17 & -3 \\
0 & -3 & -5 & 13 & 3 \\
0 & 3 & -6 & 15 & 0
\end{bmatrix}
$$

no "easy" elem. row op. using R_2 but by switching R_2 and R_3 there will be an "easy" elem. row. op.

$$
\sim \quad R_2 \Leftrightarrow R_3 \quad
\begin{bmatrix}
2 & -1 & 1 & -3 & 1 \\
0 & -3 & -5 & 13 & 3 \\
0 & 9 & -7 & 17 & -3 \\
0 & 3 & -6 & 15 & 0
\end{bmatrix}
$$

$$\sim \begin{array}{r} \\ \\ 3R_2+R_3\to R_3 \\ R_2+R_4\to R_4 \end{array}\begin{bmatrix} 2 & -1 & 1 & -3 & 1 \\ 0 & -3 & -5 & 13 & 3 \\ 0 & 0 & -22 & 56 & 6 \\ 0 & 0 & -11 & 28 & 3 \end{bmatrix}$$

no "easy" elem. row. op. but by mult. R_4 by 2 there will be an "easy" elem. row. op. using R_3 since $(2)(-11) = -22$

$$\sim \begin{array}{r} \\ \\ \\ 2R_4 \end{array}\begin{bmatrix} 2 & -1 & 1 & -3 & 1 \\ 0 & -3 & -5 & 13 & 3 \\ 0 & 0 & -22 & 56 & 6 \\ 0 & 0 & -22 & 56 & 6 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ \\ -R_3+R_4\to R_4 \end{array}\begin{bmatrix} 2 & -1 & 1 & -3 & 1 \\ 0 & -3 & -5 & 13 & 3 \\ 0 & 0 & -22 & 56 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ \frac{1}{2}R_3 \\ \\ \end{array}\begin{bmatrix} 2 & -1 & 1 & -3 & 1 \\ 0 & -3 & -5 & 13 & 3 \\ 0 & 0 & -11 & 28 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

no "easy" elem-row op. to eliminate -5 & 1 using R_3. $\therefore$ we mult R_1 & R_2 by 11

$$\sim \begin{array}{r} 11R_1 \\ 11R_2 \\ \\ \\ \end{array}\begin{bmatrix} 22 & -11 & 11 & -33 & 11 \\ 0 & -33 & -55 & 143 & 33 \\ 0 & 0 & -11 & 28 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} R_3 + R_1 \to R_1 \\ 5R_3 + R_2 \to R_2 \end{array} \begin{bmatrix} 22 & -11 & 0 & -5 & 14 \\ 0 & -33 & 0 & 3 & 18 \\ 0 & 0 & -11 & 28 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} \\ -\frac{1}{3}R_2 \\ \\ \\ \end{array} \begin{bmatrix} 22 & -11 & 0 & -5 & 14 \\ 0 & 11 & 0 & -1 & -6 \\ 0 & 0 & -11 & 28 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} R_2 + R_1 \to R_1 \\ \\ \\ \\ \end{array} \begin{bmatrix} 22 & 0 & 0 & -6 & 8 \\ 0 & 11 & 0 & -1 & -6 \\ 0 & 0 & -11 & 28 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} \frac{1}{22}R_1 \\ \frac{1}{11}R_2 \\ \frac{-1}{11}R_3 \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \frac{-3}{11} & \frac{4}{11} \\ 0 & 1 & 0 & \frac{-1}{11} & \frac{-6}{11} \\ 0 & 0 & 1 & \frac{-28}{11} & \frac{-3}{11} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_4 is a free variable. $\therefore$ $x_4 = t$ ⊛

$$\begin{cases} x_1 - \frac{3}{11}x_4 = \frac{4}{11} \\ x_2 - \frac{1}{11}x_4 = \frac{-6}{11} \\ x_3 - \frac{28}{11}x_4 = \frac{-3}{11} \end{cases} \quad \therefore$$

sub ⊛

$$\begin{cases} x_1 - \frac{3}{11}t = \frac{4}{11} \\ x_2 - \frac{1}{11}t = \frac{-6}{11} \\ x_3 - \frac{28}{11}t = \frac{-3}{11} \end{cases}$$

$$x_1 = \frac{4}{11} + \frac{3}{11}t$$

$$x_2 = \frac{-6}{11} + \frac{1}{11}t$$

$$x_3 = \frac{-3}{11} + \frac{28}{11}t$$

$$x_4 = t$$