Dawson College: **Linear Algebra**: 201-105-DW-S05: Fall 2009

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Test 1

This test is graded out of 49 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & + & x_3 & & & = & 3 \\ 4x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & = & 2 \\ 5x_1 & + & 5x_2 & + & 5x_3 & + & x_4 & = & 3 \\ 7x_1 & + & 4x_2 & + & 4x_3 & + & x_4 & = & 5 \end{array}$$

$$\begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 4 & 3 & 3 & 1 & 2 \\ 5 & 5 & 5 & 1 & 3 \\ 7 & 4 & 4 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ 3R_2 \\ 3R_3 \\ 3R_4 \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 12 & 9 & 9 & 3 & 6 \\ 15 & 15 & 15 & 3 & 9 \\ 21 & 12 & 12 & 3 & 15 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ -4R_1+R_2 \to R_2 \\ -5R_1+R_3 \to R_3 \\ -7R_1+R_4 \to R_4 \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 0 & 5 & 5 & 3 & -6 \\ 0 & 10 & 10 & 3 & 6 \\ 0 & 5 & 5 & 3 & -6 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ -2R_2+R_3 \to R_3 \\ -R_2+R_4 \to R_4 \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 0 & 5 & 5 & 3 & -6 \\ 0 & 0 & 0 & -6 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ -\frac{1}{6}R_3 \\ \\ \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 0 & 5 & 5 & 3 & -6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ -3R_3+R_2 \to R_2 \\ \\ \\ \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \frac{1}{5}R_2 \\ \\ \\ \end{array} \begin{bmatrix} 3 & 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim -R_2+R_1 \to R_1 \begin{bmatrix} 3 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \frac{1}{3}R_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable hence $x_3 = t$.

$x_1 = 1$

$x_2 + x_3 = 0 \Rightarrow x_2 = -t$

$x_4 = -2$

$\therefore$

$x_1 = 1$

$x_2 = -t$

$x_3 = t$

$x_4 = -2$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \end{bmatrix}\ C = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \end{bmatrix}\ D = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

BD

b. *(4 marks)* Compute the following, if possible.

$CB - 2A$

c. *(4 marks)* Compute the following, if possible.

$\mathrm{tr}(C^T B^T)$

d. *(4 marks)* Find E, if possible.

$(I - E^{-1})^T = D$

$$\underset{2\times 3\ \ 2\times 2}{B\ D}$$

undefined since

b)
$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 7 \\ -4 & 6 & -9 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 4 \\ 2 & 0 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & -7 \\ -2 & -2 & 1 \\ -8 & 0 & -11 \end{bmatrix}$$

c) $C^T B^T$

$$= \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 8 \\ 9 & -5 \end{bmatrix}$$

$\therefore\ tr(C^T B^T) = -10$

d)
$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - E^{-1}\right]^t = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - E^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^t$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - E^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$-E^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-E^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$E = \frac{1}{-2}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Question 3. *(5 marks)* Given the following augmented matrix in row-echelon form, solve the system using back substitution.

$$\left[\begin{array}{cccc|c} 1 & 5 & 2 & 1 & \sqrt{5} \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

X_2 is a free variable $\therefore X_2 = t$ (4)

$\Updownarrow$

$X_1 + 5X_2 + 2X_3 + X_4 = \sqrt{5}$ (5)

$X_3 + X_4 = 2$ (2)

$X_4 = 3$ (1)

sub (1) into (2)

$X_3 + 3 = 2$

$X_3 = -1$ (3)

sub (1), (3), (4) into (5)

$X_1 + 5t + 2(-1) + 3 = \sqrt{5}$

$X_1 = \sqrt{5} - 1 - 5t$

$\therefore X_1 = \sqrt{5} - 1 - 5t$

$X_2 = t$

$X_3 = -1$

$X_4 = 3$

Question 4. *(3 marks)* Find A^{-1} if $A^2 - 2A - I = 0$.

notice $A^2 - 2A = I$

$A(A - 2I) = I$

and $A^2 - 2A = I$

$(A - 2I)A = I$

$\therefore A^{-1} = A - 2I$

Question 5. *(8 marks)* Solve the following system by inverting the coefficient matrix.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ -2x_1 & - & 3x_2 & + & 4x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 10x_3 & = & -1 \end{array}$$

$$Ax = b$$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & 4 \\ 3 & 6 & 10 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Lets find the inverse

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -3 & 4 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{2R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 10 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{-3R_3+R_1 \to R_1 \\ -10R_3+R_2 \to R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 10 & 0 & -3 \\ 0 & 1 & 0 & 32 & 1 & -10 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\underset{-2R_2+R_2 \to R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -54 & -2 & 17 \\ 0 & 1 & 0 & 32 & 1 & -10 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$A^{-1}$$

$$\therefore \quad Ax = b$$
$$x = A^{-1}b$$

$$x = \begin{bmatrix} -54 & -2 & 17 \\ 32 & 1 & -10 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -71 \\ 42 \\ -4 \end{bmatrix}$$

Question 6. Consider the matrices

$$A = \begin{bmatrix} 3 & 5 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$$

Find the elementary matrices E_1, E_2 and E_3 (if possible) such that

a. *(2 marks)* $E_1 B = C$

b. *(2 marks)* $E_2 A = B$

c. *(2 marks)* $E_3 A = C$

a) $B = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 3 & 5 \\ 2 & -3 \end{bmatrix} \underset{R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} = C$

E_1 does not exists since 2 elementary row operations are required to obtain C.

b) $A = \begin{bmatrix} 3 & 5 \\ 2 & -3 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix} = B$

$\therefore E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 5 \\ 2 & -3 \end{bmatrix} \underset{R_1 + R_2 \to R_2}{\sim} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} = C$

$\therefore E_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Question 7. *(3 marks)* Show that if A is invertible and $AB = AC$, then $B = C$.

A invertible $\Rightarrow A^{-1}A = I$

$$\therefore \quad AB = AC$$
$$A^{-1}AB = A^{-1}AC$$
$$IB = IC$$
$$B = C$$

$$\therefore \; B = C$$

Bonus Question. *(3 marks)* Consider

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, b, c \neq 0$ and find A^{-1} as a product of elementary matrices.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \sim_{\frac{1}{c}R_2} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \sim_{-bR_2 + R_1 \to R_1} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{\frac{1}{a}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore \; E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c} \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}$$

and $\underbrace{E_3 E_2 E_1}_{A^{-1}} A = I$

$$\therefore \; A^{-1} = E_3 E_2 E_1$$