Dawson College: Calculus I: 201-NYA-05 S19 September 25, 2012

Last Name: SOLUTIONS

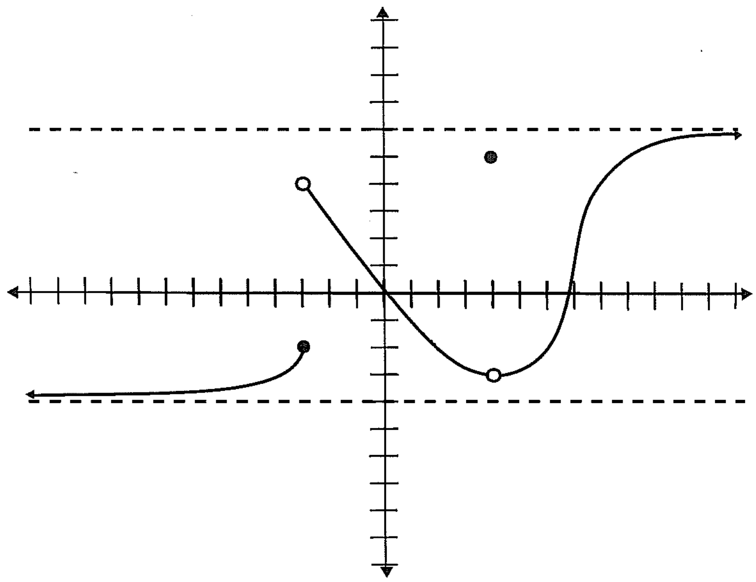
First Name: ______________

Student ID: ______________

Test 1

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Circle your last name for one bonus mark. Do not use decimals unless otherwise stated.

Question 1. *(0.5 points each)* The graph of $f(x)$ is below (each hash mark is 1 unit). Use the graph to find/answer the following (just state the answer, you do not need to show any work):



(a) $\lim_{x \to -3^-} f(x) = -2$

(b) $\lim_{x \to -3^+} f(x) = 4$

(c) $\lim_{x \to 0} f(x) = 0$

(d) $\lim_{x \to 2} f(x) = -2$

(e) $f(2) = -2$

(f) $\lim_{x \to -3} f(x)$ D.N.E.

(g) Is $f(x)$ continuous at $x = -3$?

NO

(h) Is $f(x)$ continuous at $x = 0$?

YES

(i) Is $f(x)$ continuous at $x = 2$?

YES

(j) $\lim_{x \to \infty} f(x) = 6$

(k) $\lim_{x \to -\infty} f(x) = -4$

(l) Is $f(x)$ left continuous or right continuous at $x = -3$?

LEFT CONTINUOUS.

Question 2. Find the following limits:

(a) *(2 marks)* $\lim_{x\to 3} \frac{x^2-x-6}{2x^2-x-3}$ $= \frac{(3)^2-3-6}{2(3)^2-3-3} = \frac{0}{12} = 0$

(b) *(4 marks)* $\lim_{x\to 2} \frac{4-\sqrt{5x+6}}{x-2} \cdot \frac{4+\sqrt{5x+6}}{4+\sqrt{5x+6}} = \lim_{x\to 2} \frac{16-(5x+6)}{x-2(4+\sqrt{5x+6})}$

$= \lim_{x\to 2} \frac{10-5x}{x-2(4+\sqrt{5x+6})} = \lim_{x\to 2} \frac{5(2-x)}{(x-2)(4+\sqrt{5x+6})}$

$= \lim_{x\to 2} \frac{-5}{4+\sqrt{5x+6}} = \frac{-5}{4+\sqrt{16}} = \frac{-5}{8}$

(c) *(4 marks)* $\lim_{x\to 1^+} \frac{|1-x|}{x^2-1} \overset{*}{=} \lim_{x\to 1^+} \frac{-(1-x)}{(x+1)(x-1)} = \lim_{x\to 1^+} \frac{1}{x+1}$

$= \frac{1}{1+1} = \frac{1}{2}$

$*$ IF $x \to 1^+ \Rightarrow x > 1 \Rightarrow 0 > 1-x \Rightarrow |1-x| = -(1-x)$

(d) *(4 marks)* $\lim_{\theta\to 0} \dfrac{\sin(2\theta)}{\sin(3\theta)\cdot\cos\theta}$ $= \cancel{\lim_{\theta\to 0} \dfrac{\sin 2\theta}{2\theta} \cdot \left(\dfrac{1}{\frac{\sin 3\theta}{3\theta}}\right) \cdot \dfrac{1}{\cos\theta}} \cdot$

$= \lim_{\theta\to 0} \dfrac{\sin 2\theta}{2\theta} \cdot \dfrac{3\theta}{\sin 3\theta} \cdot \dfrac{1}{\cos\theta} \cdot \dfrac{2}{3}$

$= (1)(1)(1)\cdot\dfrac{2}{3}$

$= \dfrac{2}{3}$

(e) *(4 marks)* $\lim_{x\to\infty} \dfrac{4x^{3/2}+3\sqrt{x}-5}{2x^{3/2}-7x+6}$ $= \lim_{x\to\infty} \dfrac{\frac{4x^{3/2}}{x^{3/2}} + \frac{3x^{1/2}}{x^{3/2}} - 5/x^{3/2}}{\cancel{\frac{2x^{3/2}}{x^{3/2}}} - \frac{7x}{x^{3/2}} + 6/x^{3/2}}$

$= \lim_{x\to\infty} \dfrac{4 + 3/x - 5/x^{3/2}}{2 - 7/x^{1/2} + 6/x^{3/2}}$ $= \dfrac{4+0-0}{2-0+0} = 2$

(f) *(3 marks)* $\lim_{x\to\infty} \dfrac{5x^5-2}{-3x+6}$ $= \lim_{x\to\infty} \dfrac{5x^5/x - 2/x}{\frac{-3x}{x} + 6/x}$

$= \lim_{x\to\infty} \dfrac{\overset{\to\infty}{5x^4} - \overset{\to 0}{2/x}}{\underset{-3}{-3} + \underset{\to 0}{6/x}}$ $= -\infty$

Question 3. *(5 marks)* Find the value of c that would make the following function continuous everywhere.

$$f(x) = \begin{cases} x^2 - cx - 5 & \text{if } x < 1 \\ -2 & \text{if } x = 1 \\ c^2x + 3c & \text{if } x > 1 \end{cases}$$

1) $f(1) = -2$

2) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - cx - 5) = (1)^2 - c(1) - 5 = -c - 4$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (c^2 x + 3c) = c^2 + 3c$

$\therefore$ FOR $\lim_{x \to 1} f(x)$ TO EXIST

$-c - 4 = c^2 + 3c \Rightarrow 0 = c^2 + 4c + 4$

$0 = (c+2)(c+2) \Rightarrow c = -2.$

NOW IF $c = -2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 - (-2)x - 5 = (1)^2 + 2(1) - 5$

$= 3 - 5 = -2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-2)^2 x + 3(-2) = 4(1) - 6 = -2$

SO $\lim_{x \to 1} f(x) = -2$

3) $\lim_{x \to 1} f(x) = -2 = f(1)$ (IF $c = -2$)

$\therefore$ IF $c = -2$ f IS CONTINUOUS AT $x = 1$

BUT f IS CONTINUOUS EVERYWHERE ELSE SINCE IT IS A POLYNOMIAL

SO IF $c = -2$ f IS CONTINUOUS EVERYWHER.

Question 4. Determine if each of the following functions has a removable discontinuity at the given x-value:

(a) *(3 marks)* $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$ at $x = 3$

$$\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9} = \lim_{x \to 3} \frac{x(x-3)}{(x+3)(x-3)} = \lim_{x \to 3} \frac{x}{x+3} = \frac{3}{3+3} = \frac{1}{2}$$

BUT $f(3)$ IS NOT DEFINED.

$\therefore$ $f(x)$ HAS A REMOVABLE DISCONTINUITY AT $x = 3$.

(b) *(3 marks)* $f(x) = [[\cos x]]$ at $x = 0$

$\lim_{x \to 0^-} [[\cos x]] = 0$ $\qquad$ $\lim_{x \to 0^+} [[\cos x]] = 0$

IF $-\frac{\pi}{2} < x < 0 \Rightarrow 0 < \cos x < 1$ $\qquad$ IF $0 < x < \pi/2 \Rightarrow 0 < \cos x < 1$

$\therefore \lim_{x \to 0} [[\cos x]] = 0$

BUT $f(0) = [[\cos(0)]] = [[1]] = 1$

SO $\lim_{x \to 0} [[\cos x]] \neq f(0)$

$\therefore$ f HAS A REMOVABLE DISCONTINUITY AT $x = 0$.

Question 5. *(5 marks)* Use the squeeze theorem to show that:

$$\lim_{x \to 0} x^6 \cos^2\left(1+\frac{1}{x}\right) = 0$$

NOTICE

$$0 \le \cos^2\left(1+\frac{1}{x}\right) \le 1$$

SINCE $x^6 \ge 0$ $\Rightarrow$ $x^6 \cdot 0 \le \cos^2\left(1+\frac{1}{x}\right) \le x^6$

NOW $\lim_{x \to 0} (0) = 0$ AND $\lim_{x \to 0} (x^6) = 0$

SO BY SQUEEZE THEOROM

$$\lim_{x \to 0} x^6 \cos^2\left(1+\frac{1}{x}\right) = 0$$

Question 6. *(4 marks)* Suppose $f(x)$ is continuous at $x = 4$ and $f(2) = 2$ and find

$$\lim_{x\to 4} \frac{[f(x)]^2 + 3f(x) - 10}{[f(x)]^2 - 4}$$

(Clearly show all of your steps.)

$$\lim_{x\to 4} \frac{[f(x)]^2 + 3f(x) - 10}{[f(x)]^2 - 4} = \lim_{x\to 4} \frac{(f(x) - 2)(f(x) + 5)}{(f(x) - 2)(f(x) + 2)}$$

$$= \lim_{x\to 4} \frac{f(x) + 5}{f(x) + 2} = \frac{\lim_{x\to 4} (f(x) + 5)}{\lim_{x\to 4} (f(x) + 2)}$$

SINCE f IS CONTINUOUS AT $x = 4$

$$= \frac{\lim_{x\to 4} f(x) + 5}{\lim_{x\to 4} f(x) + 2} = \frac{f(4)) + 5}{f(4) + 2}$$

$$= \frac{2 + 5}{2 + 2} = \frac{7}{4}$$

Bonus Question. *(4 marks)* Let $f(x)$ and $g(x)$ be continuous funtions (everywhere). Suppose that $g(1) < f(1)$ and $g(6) > f(6)$. Show that there is a number c such that $f(c) = g(c)$. (Hint: define a new function $h(x) = g(x) - f(x)$. Does this function have any roots?)

LET $h(x) = g(x) - f(x)$. SINCE $g(x)$ AND $f(x)$ ARE CONTINUOUS SO IS $h(x)$. IN PARTICULAR, $h(x)$ IS CONTINUOUS ON $[1, 6]$

NOW $h(1) = g(1) - f(1) < 0$ SINCE $g(1) < f(1)$

AND $h(6) = g(6) - f(6) > 0$ SINCE $g(6) > f(6)$

SO $h(1) < 0 < h(6)$

$\therefore$ BY THE INTERMEDIATE VALUE THEOREM THERE IS A c IN $(1, 6)$ SUCH THAT

$h(c) = 0$ BUT THIS MEANS THAT

$0 = h(c) = g(c) - f(c) \Rightarrow f(c) = g(c)$

SO THERE IS A c IN $(1, 6)$ SUCH THAT $f(c) = g(c)$ ∎