Last Name: SOLUTIONS

First Name:

Student ID:

Test 2

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

Question 1. *(5 marks)* Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x^2+1}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} = \lim_{h\to 0} \frac{\sqrt{x^2+2xh+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x^2+2xh+h^2+1} - \sqrt{x^2+1}}{h} \cdot \frac{\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}}{\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}}$$

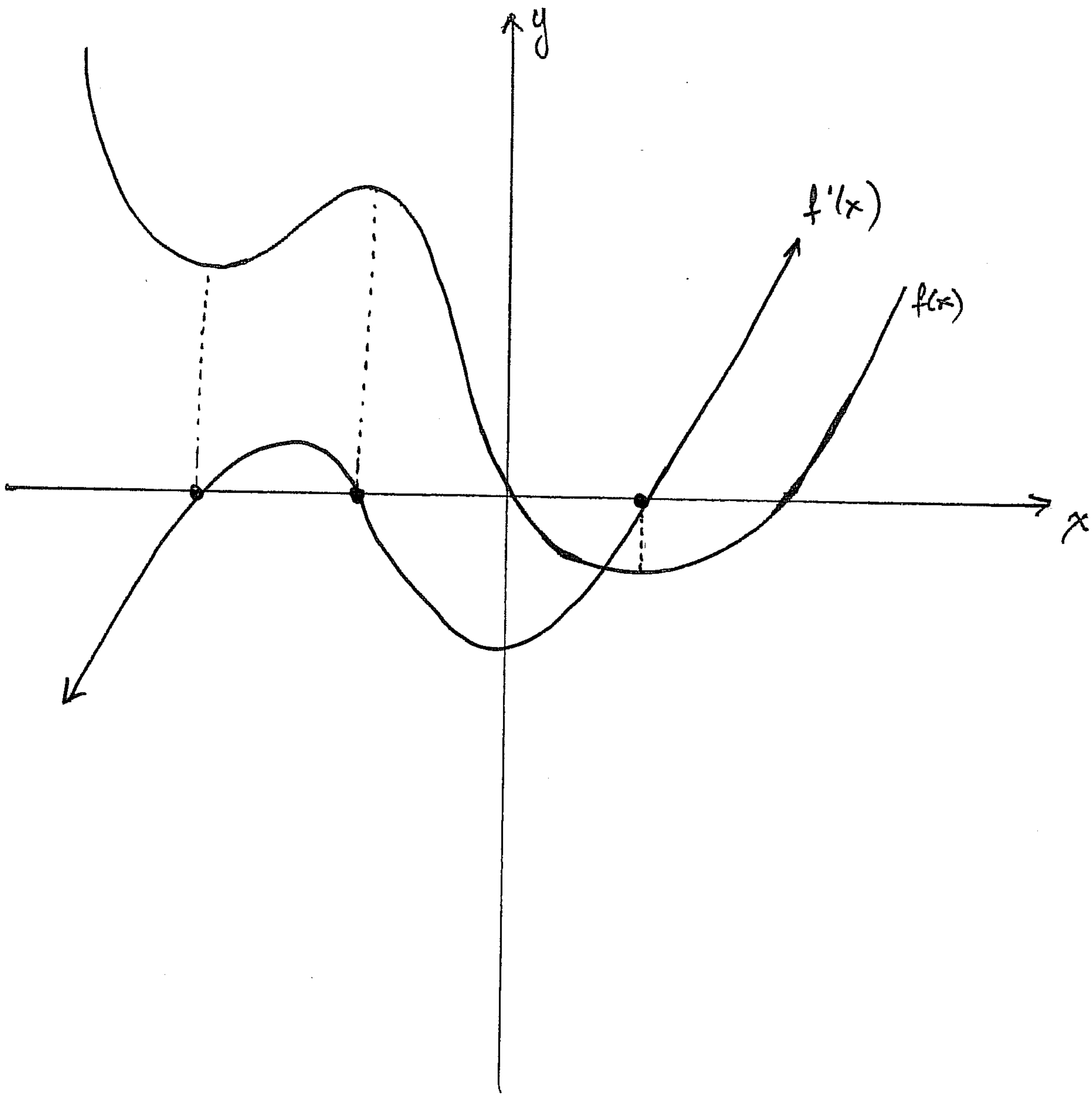
$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2+1)-(x^2+1)}{h\left(\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{x^2+2xh+h^2+1} - \sqrt{x^2+1}}$$

$$= \frac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

Question 2. *(3 marks)* Sketch the graph of $f'(x)$ on top of the graph of $f(x)$ given below.



Question 3. Find the derivatives of the following functions:

(a) *(3 marks)* $y = x^3 - 2x^2 + \sqrt[3]{x^2} - \pi$ $= x^3 - 2x^2 + x^{2/3} - \pi$

$y' = 3x^2 - 4x + \frac{2}{3} x^{-1/3} - 0$

$= 3x^2 - 4x + \frac{2}{3x^{1/3}}$

(b) *(4 marks)* $g(x) = \sqrt{\frac{x^4+2x-2}{3x^3-4x+7}}$ $= \left(\frac{x^4+2x-2}{3x^3-4x+7}\right)^{1/2}$

$g'(x) = \frac{1}{2}\left(\frac{x^4+2x-2}{3x^3-4x+7}\right)^{-1/2} \cdot \frac{(4x^3+2)(3x^3-4x+7) - (x^4+2x-2)(9x^2-4)}{(3x^3-4x+7)^2}$

(c) *(5 marks)* $f(\theta) = \sec^3(4\theta^3+3)$ $= [\sec(4\theta^3+3)]^3$

$$f'(\theta) = 3[\sec(4\theta^3+3)]^2 \cdot \frac{d}{d\theta}\left(\sec(4\theta^3+3)\right)$$

$$= 3[\sec(4\theta^3+3)]^2 \sec(4\theta^3+3)\tan(4\theta^3+3)\cdot(12\theta^2)$$

(d) *(4 marks)* $h(x) = \sin x \csc(\tan x)$

$$h'(x) = \cos x \cdot \csc(\tan x) + \sin x \cdot \left(-\csc(\tan x)\cot(\tan x)\right) \cdot \sec^2 x$$

Question 4. *(5 marks)* Find all x-values where the tangent line to the graph of $f(x)$ is horizontal.

$$f(x) = \frac{(4x+1)^3}{(2x-3)^4}$$

$$f'(x) = \frac{3(4x+1)^2 \cdot 4 \cdot (2x-3)^4 - (4x+1)^3 \cdot 4(2x-3)^3 \cdot 2}{[(2x-3)^4]^2}$$

$$= \frac{4(4x+1)^2(2x-3)^3[3(2x-3) - 2(4x+1)]}{(2x-3)^8}$$

$$= \frac{4(4x+1)^2(6x-9-8x-2)}{(2x-3)^5}$$

$$= \frac{4(4x+1)^2(-2x-11)}{(2x-3)^5} = 0$$

$\Leftrightarrow \quad 4x+1=0 \quad$ or $\quad -2x-11=0$

$\Rightarrow \quad x = -1/4 \quad$ or $\quad x = -11/2$

$\therefore$ THE TANGENT LINE TO f IS HORIZONTAL AT

$x = -1/4$ AND $x = -\frac{11}{2}$

Question 5. *(4 marks)* Use the quotient rule to show that

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{-\sin x \cdot \sin x - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

Question 6. *(4 marks)* The function $H(x)$ is defined by $H(x) = g(x^2 + f(x))$. Given that $f(2) = 3$, $f'(2) = 5$ and $g'(7) = -2$ find $H'(2)$

BY CHAIN RULE

$$H'(x) = g'(x^2+f(x)) \cdot \frac{d}{dx}\left[x^2+f(x)\right]$$

$$= g'(x^2+f(x)) \cdot (2x + f'(x))$$

SO $H'(2) = g'((2)^2+f(2)) \cdot (2(2)+f'(2))$

$$= g'(4+3) \cdot (4+5)$$

$$= g'(7) \cdot 9$$

$$= (-2) \cdot 9$$

$$= -18$$

Question 7.

(a) *(5 marks)* Given $x^2+y^2-x=\sqrt{xy}$ find $\frac{dy}{dx}$.

$$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]-\frac{d}{dx}[x]=\frac{d}{dx}[\sqrt{xy}]$$

$$2x+2y\frac{dy}{dx}-1=\frac{1}{2}(xy)^{-1/2}\cdot\left(1\cdot y+x\frac{dy}{dx}\right)$$

$$2x+2y\frac{dy}{dx}-1=\frac{1}{2}(xy)^{-1/2}\cdot y+\frac{1}{2}x(xy)^{-1/2}\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{1}{2}x(xy)^{-1/2}\frac{dy}{dx}=\frac{1}{2}y(xy)^{-1/2}-2x+1$$

$$\frac{dy}{dx}\left[2y-\frac{1}{2}x(xy)^{-1/2}\right]=\frac{1}{2}y(xy)^{-1/2}-2x+1$$

$$\frac{dy}{dx}=\frac{\frac{1}{2}y(xy)^{-1/2}-2x+1}{2y-\frac{1}{2}x(xy)^{-1/2}}$$

(b) *(3 marks)* Find the tangent line to the curve $x^2+y^2-x=\sqrt{xy}$ at the point $(1,1)$

AT $(1,1)$

$$\frac{dy}{dx}=\frac{\frac{1}{2}(1)(1\cdot1)^{-1/2}-2(1)+1}{2(1)-\frac{1}{2}(1)(1\cdot1)^{-1/2}}=\frac{\frac{1}{2}-2+1}{2-\frac{1}{2}}=\frac{-\frac{1}{2}}{\frac{3}{2}}=-\frac{1}{3}$$

$\therefore\quad m=-\frac{1}{3}\quad\Rightarrow\quad y=mx+b$

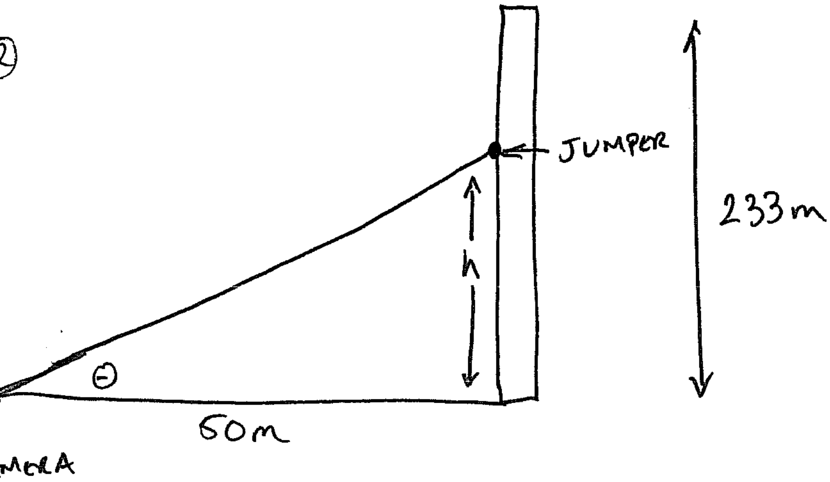
$$1=-\frac{1}{3}(1)+b$$

$$\frac{4}{3}=b$$

$$\therefore\quad \boxed{y=-\frac{1}{3}x+\frac{4}{3}}$$

Question 8. *(5 marks)* A bungee jumper jumps from the Macau Tower 233 metres from the ground (the highest commercial bungee jump). She is falling at a rate of 40m/s when she has fallen 83m or, in other words, when she is 150m from the ground. A camera person stands 50 metres from the base of the tower and films the jump. If the camera is always focused on the jumper, how fast is the camera's angle of elevation decreasing when the jumper has fallen 83m?

2)



(3) θ, h ARE FUNCTIONS OF t.

(4) $\frac{dh}{dt} = -40\,m/s$, $\frac{d\theta}{dt} = ?$ WHEN $h = 150m$

(5) $\tan\theta = \frac{h}{50}$

(6) $\frac{d}{dt}[\tan\theta] = \frac{d}{dt}\left[\frac{h}{50}\right]$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{50}\frac{dh}{dt}$

(7) $\tan\theta = \frac{150}{50} = 3$ WHEN $h = 150m$

SINCE $\sec^2\theta = \tan^2\theta + 1 = 3^2 + 1 = 10$ WE GET

$10\frac{d\theta}{dt} = \frac{1}{50}\cdot(-40) \Rightarrow \frac{d\theta}{dt} = -0.08$ rad/s

(8) THE CAMERA'S ANGLE OF ELEVATION IS DECREASING AT A RATE OF 0.08 rad/s WHEN THE JUMPER HAS FALLEN 83m.

Bonus Question. *(3 marks)* The **Bessel Function** of order 0, $y = J(x)$, satisfies the equation $xy'' + y' + xy = 0$ for all values of x. Given $J(0) = J'(0) = 0$ find $J''(0)$.

$$\frac{d}{dx}[xy''] + \frac{d}{dx}[y'] + \frac{d}{dx}[xy] = 0$$

$$1 \cdot y'' + xy''' + y'' + 1 \cdot y + xy' = 0$$

$$\Rightarrow \quad J''(0) + 0 \cdot J'''(0) + J''(0) + J(0) + 0 \cdot J'(0) = 0$$

$$J''(0) + J''(0) + 0 = 0$$

$$2J''(0) = 0$$

$$J''(0) = 0$$