Dawson College: Calculus I: 201-NYA-05 S19 November 26, 2012

Last Name: SOLUTIONS

First Name: ______________

Student ID: ______________

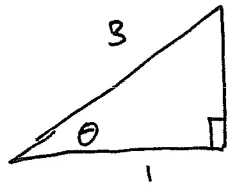
Test 3

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

Question 1. Evaluate the following expressions without using a calculator (show your work)

(a) *(2 marks)* $\sin(\cos^{-1}(1/3))$

LET $\theta = \cos^{-1}\left(\frac{1}{3}\right) \Rightarrow$



$\sqrt{3^2-1^2} = \sqrt{8} = 2\sqrt{2}$

$\therefore \quad \sin\left(\cos^{-1}(1/3)\right) = \sin\theta = \dfrac{2\sqrt{2}}{3}$

(b) *(2 marks)* $2\log_3 10 - \log_3 18 - \log_3 50 \Rightarrow \log_3 10^2 - \log_3 18 - \log_3 50$

$\Rightarrow \log_3 \frac{100}{18} - \log_3 50 = \log_3 \frac{100}{18 \cdot 50} = \log_3 \frac{2}{18} = \log_3 \frac{1}{9}$

$= -2$

Question 2. Find the following derivatives

(a) *(4 marks)* $f(x) = 2^{\arctan x^2}$

$$f'(x) = 2^{\arctan x^2} \cdot \ln 2 \cdot \frac{1}{1+(x^2)^2} \cdot 2x$$

(b) *(4 marks)* $y = e^{x^2} \cdot \log_5(\sin^{-1} x)$

$$y' = 2xe^{x^2} \cdot \log_5(\sin^{-1}x) + e^{x^2} \frac{1}{\sin^{-1}x \cdot \ln 5} \cdot \frac{1}{\sqrt{1-x^2}}$$

(c) *(5 marks)* $y = (\sin x)^{\ln x}$

$$\ln y = \ln x \cdot \ln \sin x$$

$$\frac{1}{y} \cdot y' = \frac{1}{x} \ln(\sin x) + \ln x \cdot \frac{1}{\sin x} \cdot (+\cos x)$$

$$y' = y\left[\frac{1}{x} \ln(\sin x) + \ln x \cdot \cot x\right]$$

$$= (\sin x)^{\ln x} \left[\frac{1}{x} \ln(\sin x) + \ln x \cdot \cot x\right]$$

Question 3. Evaluate the following limits:

(a) *(4 marks)* $\lim_{x \to 0} \frac{\cos x - \cos 2x}{e^x - 1 - x}$ $= \text{I.F. } \frac{0}{0} \overset{H}{=} \lim_{x \to 0} \frac{-\sin x + 2\sin 2x}{e^x - 1}$

$= \text{I.F. } \frac{0}{0} \overset{H}{=} \lim_{x \to 0} \frac{-\cos x + 4\cos 2x}{e^x}$

$= \frac{-\cos(0) + 4\cos(0)}{e^0} = \frac{-1+4}{1} = 3$

(b) *(5 marks)* $\lim_{x \to \infty} (e^x + 1)^{1/x}$ LET $y = (e^x+1)^{1/x}$

$\ln y = \frac{1}{x} \ln(e^x+1)$

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{1}{x} \ln(e^x+1) = \text{I.F. } 0 \cdot \infty$

$= \lim_{x \to \infty} \frac{\ln(e^x+1)}{x} = \text{I.F. } \frac{\infty}{\infty} \overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{e^x+1} \cdot e^x}{1}$

$= \lim_{x \to \infty} \frac{e^x}{e^x+1} = \text{I.F. } \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{e^x}{e^x} = \lim_{x \to \infty} 1 = 1$

$\therefore \lim_{x \to \infty} (e^x+1)^{1/x} = \lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln y} = e^{\lim_{x \to \infty} \ln y}$

$= e^1 = e$

Question 4. *(4 marks)* Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \leq x \leq 2$.

$$f'(x) = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$= 3(x-3)(x+1) = 0$$
$$\therefore \quad x = 3, -1 \qquad \text{BUT } x = 3 \notin [-2, 2]$$

SO $x = -1$ IS THE ONLY C.N.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = 7$$
$$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = 0$$
$$f(2) = (2)^3 - 3(2)^2 - 9(2) + 2 = -20$$

$\therefore$ f HAS AN ABSOLUTE MAXIMUM $f(-1) = 7$ AND ABSOLUTE MINIMUM $f(2) = -20$ ON $[-2, 2]$

Question 5. *(4 marks)* Use the second derivative test to find the local maximum and loval minimum values of $f(x) = x^2e^{-x}$.

$$f'(x) = 2xe^{-x} + x^2(-e^{-x})$$
$$= e^{-x}(2x - x^2) = 0$$

$$\Rightarrow \quad 2x - x^2 = x(2-x) = 0$$

$$\therefore x = 0, 2$$

$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2-2x)$$

$$= e^{-x}[-(2x-x^2) + (2-2x)]$$

$$= e^{-x}[-2x + x^2 + 2 - 2x] = e^{-x}(x^2 - 4x + 2)$$

$$f''(0) = e^0(0^2 - 4(0) + 2) = 2 > 0$$

$\therefore f(0) = 0^2e^{-0} = 0$ IS A LOCAL MINIMUM

$$f''(2) = e^{-2}(2^2 - 4(2) + 2) = e^{-2}(4 - 8 + 2) = -2e^{-2} < 0$$

$\therefore f(2) = 4e^{-2}$ IS A LOCAL MAXIMUM.

Question 6. Given

$$f(x) = \frac{x^2-1}{x^2-4} \qquad f'(x) = \frac{-6x}{(x^2-4)^2} \qquad f''(x) = \frac{6(3x^2+4)}{(x^2-4)^3}$$

Find:

(a) *(3 marks)* The domain and the x and y-intercepts of f

DOMAIN: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

x-int: $y=0$

$$0 = \frac{x^2-1}{x^2-4}$$

$$0 = x^2-1$$

$\therefore\ x=\pm 1 \Rightarrow (1,0)$ AND $(-1,0)$

y-int: $x=0$

$$y = \frac{0^2-1}{0^2-4} = \frac{1}{4}$$

$\therefore\ \left(0,\frac{1}{4}\right)$

(b) *(4 marks)* Any horizontal and vertical asymptotoes.

$$\lim_{x\to\infty}\frac{x^2-1}{x^2-4} = \lim_{x\to\infty}\frac{1-1/x^2}{1-4/x^2} = \frac{1-0}{1-0} = 1 \quad \therefore\ y=1 \text{ IS A H.A.}$$

$$\lim_{x\to-\infty}\frac{x^2-1}{x^2-4} = \lim_{x\to-\infty}\frac{1-1/x^2}{1-4/x^2} = \frac{1-0}{1-0} = 1 \quad \therefore\ y=1 \text{ IS A H.A. (AGAIN)}$$

$$\lim_{x\to-2^-}\left(\frac{x^2-1}{x^2-4}\right) = \text{“}\frac{3}{0^+}\text{”} = \infty$$

$$\lim_{x\to-2^+}\left(\frac{x^2-1}{x^2-4}\right) = \text{“}\frac{3}{0^-}\text{”} = -\infty$$

$x=-2$ IS A V.A.

$$\lim_{x\to2^-}\frac{x^2-1}{x^2-4} = \text{“}\frac{3}{0^-}\text{”} = -\infty$$

$$\lim_{x\to2^+}\frac{x^2-1}{x^2-4} = \text{“}\frac{3}{0^+}\text{”} = \infty$$

$\therefore\ x=2$ IS A V.A.

(c) *(4 marks)* The intervals where f is increasing and where f is increasing and any relative maxima and minima.

$f'(x)=0$

$-6x=0$

$x=0$

$f'(x)$ D.N.E

$x^2-4=0$

$x=\pm 2$

TEST POINTS

$f'(-3) = \frac{18}{25} > 0$

$f'(-1) = \frac{6}{9} > 0$

$f'(1) = \frac{-1}{9}$

$f'(3) = \frac{-18}{25}$

x	-3	-2	-1	0	1	2	3
f'	$+$		$+$		$-$		$-$
f	↗		↗		↘		↘

$\therefore\ f$ IS INCREASING ON $(-\infty,-2)$ AND $(-2,0)$

f IS DECREASING ON $(0,2)$ AND $(2,\infty)$

$\therefore\ f(0) = \frac{1}{4}$ IS A RELATIVE MAXIMUM

(d) *(4 marks)* The intervals where f is concave upward and where f is concave downward and any inflection points.

$f''(x) = 0$
$6(3x^2+4) \neq 0$

$f''(x)$ D.N.E.
$(x^2-4)^3 = 0$
$x = \pm 2$

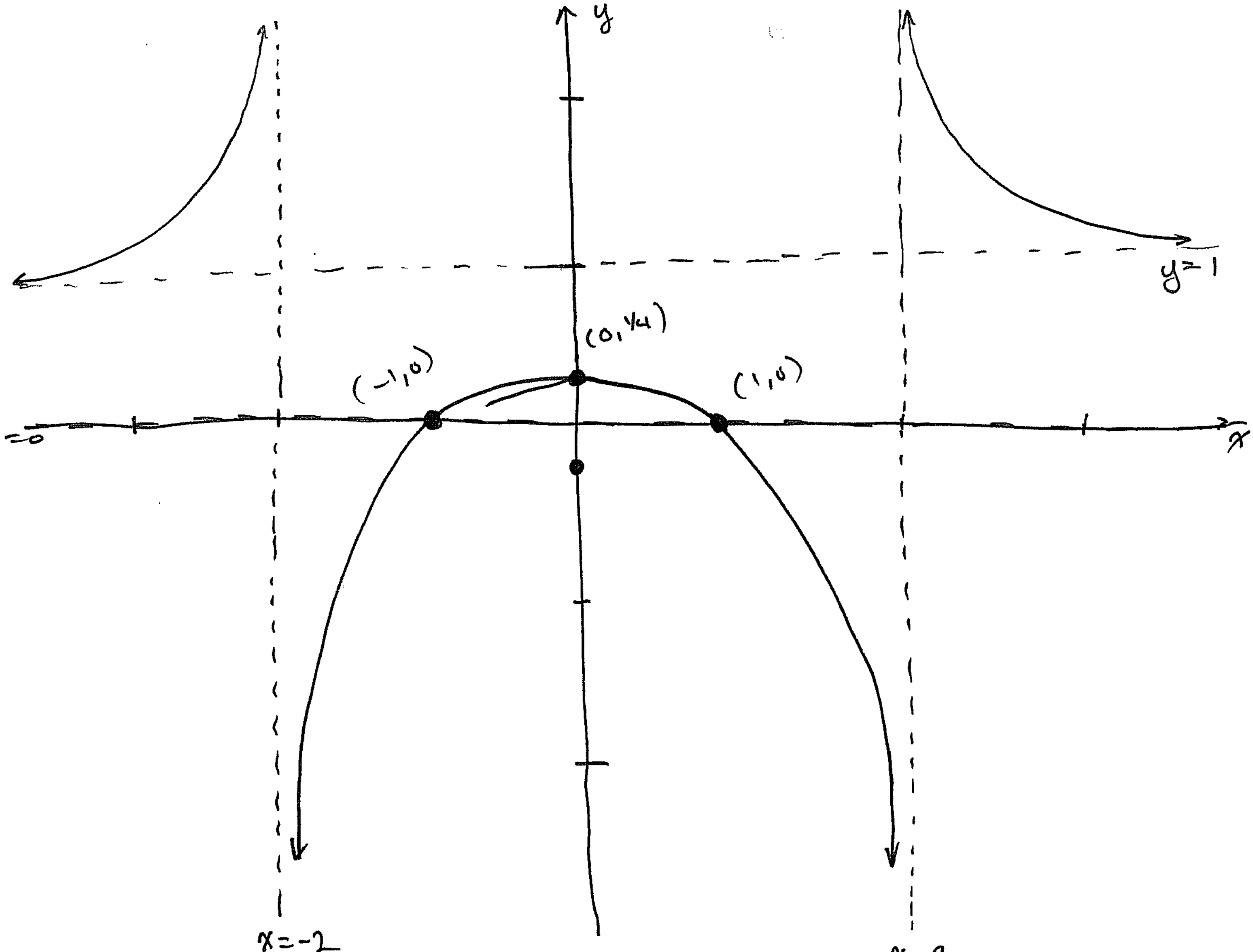
TEST POINTS
$f''(-3) = \frac{6(31)}{5^3} > 0$
$f''(0) = \frac{6 \cdot 4}{(-4)^3} < 0$
$f''(3) = \frac{6(31)}{5^3} > 0$

x	-3	-2	0	2	3
f''	$+$		$-$		$+$

f IS CONCAVE UPWARD ON $(-\infty, -2)$ AND $(2, \infty)$
f IS CONCAVE DOWNWARD ON $(-2, 2)$

NO INFLECTION POINTS.

(e) *(4 marks)* Sketch the graph of f. Label the graph with any relevant information.



Bonus Question. *(3 marks)* Suppose f is a positive function and

$$\lim_{x\to a} f(x) = 0 \quad \lim_{x\to a} g(x) = \infty$$

Show that 0^∞ is not an indeterminate form by evaluating

$$\lim_{x\to a} [f(x)]^{g(x)}$$

LET $y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x) \ln[f(x)]$

$\therefore \lim_{x\to a} \ln y = \lim_{x\to a} \underset{\downarrow \atop \infty}{g(x)} \cdot \ln[\underset{\downarrow \atop 0^+}{f(x)}] = -\infty$

SINCE $\lim_{x\to a} g(x) = \infty \quad \lim_{x\to a} \ln[f(x)] = -\infty$

SO $\lim_{x\to a} [f(x)]^{g(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$

$\therefore$ 0^∞ IS NOT AN INDETERMINATE FORM.