Dawson College: **Linear Algebra**: 201-NYC-05-S04: Fall 2014

Name: ____________________
Student ID: ____________________

Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$
\begin{array}{lrcll}
\mathscr{L}_1: & (x,y,z) & = & (2+5t, 1+t, -t) & t \in \mathbb{R} \\
\mathscr{L}_2: & (x,y,z) & = & (7+2t, 4, 10t) & t \in \mathbb{R} \\
\mathscr{L}_3: & (x,y,z) & = & (9-t, 2, 9-5t) & t \in \mathbb{R} \\
\mathscr{P}_1: & x-2y+3z-11 & = & 0 & \\
\mathscr{P}_2: & -5x-y+z+31 & = & 0 & \\
\mathscr{P}_3: & -3x+6y-9z+1 & = & 0 &
\end{array}
$$

a. *(2 marks)* Are $\mathscr{P}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify?

b. *(5 marks)* Are $\mathscr{P}_1$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify? If parallel, find the shortest distance between the two planes using projections.

c. *(3 marks)* Are $\mathscr{P}_2$ and $\mathscr{P}_3$ parallel, perpendicular, or neither, justify? Find the shortest distance between the two planes, justify.

d. *(5 marks)* Are $\mathscr{L}_2$ and $\mathscr{L}_3$ parallel, perpendicular, or neither, justify? If parallel, find the shortest distance between the two lines using projections.

Question 2. Given

$$A = \begin{bmatrix} 9 & 9 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 9 & 9 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 7 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & 2 & 6 \\ 0 & 2 & 3 & -3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -12 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad D = \begin{bmatrix} b & d \\ 3a-2b & 3c-2d \end{bmatrix}$$

a. *(4 marks)* If F is a 10×10 matrix show that AF is not invertible.

b. *(4 marks)* If E is an invertible matrix then evaluate $\det(E^{-1})^4 \det(\det(E)\,\mathrm{adj}(B))$, justify fully.

c. *(4 marks)* If $\det(D) = 2$ then determine $\det(C)$.

Question 3. If A is an $n \times n$ matrix which entries are all divisible by 2.

a. *(2 marks)* Justify that the entries of $\text{adj}(A)$ are all divisible by 2.

b. *(2 marks)* Prove using a. and $\det(A) = 2$ that the entries of A^{-1} are all integers.

Question 4. *(2 marks)* If $\vec{u}, \vec{v}$ and $\vec{w}$ be pairwise orthogonal vectors and are all of the same length show that they all make the same angle with $\vec{u} + \vec{v} + \vec{w}$.

Question 5. *(2 marks)* Prove or disprove: The general solution of the nonhomogeneous linear system $Ax = b$ can be obtaines by adding b to the general solution of the homogeneous linear system $Ax = 0$.

Question 6. *(4 marks)* Solve *only* for x_2 using Cramer's rule.

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 1 \\ x_1 & + & 3x_2 & - & 3x_3 & = & -3 \\ x_1 & - & 2x_2 & - & x_3 & = & -1 \end{array}$$

Question 7. *(3 marks)* Solve for λ.

$$\begin{vmatrix} \lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & \lambda & -6 \\ 1 & 3 & \lambda-5 \end{vmatrix}$$

Bonus Question. *(5 marks)*[1]
The Cayley-Hamilton theorem (named after the mathematicians Arthur Cayley and William Rowan Hamilton) states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

More precisely, if A is a given $n \times n$ matrix and I_n is the $n \times n$ identity matrix, then the characteristic polynomial of A is defined as

$$p(\lambda) = \det(\lambda I_n - A)$$

where det is the determinant operation. Since the entries of the matrix are (linear or constant) polynomials in λ, the determinant is also an n^{th} order polynomial in λ.

The Cayley-Hamilton theorem states that "substituting" the matrix A for λ in this polynomial results in the zero matrix,

$$p(A) = 0$$

The powers of A, obtained by substitution from powers of λ, are defined by repeated matrix multiplication; the constant term of $p(\lambda)$ gives a multiple of the power A^0, which power is defined as the identity matrix.

Prove the Cayley-Hamilton theorem for 2×2 matrices.

[1] Wikipedia contributors. "Cayley-Hamilton theorem." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 17 Oct. 2014. Web. 31 Oct. 2014.