1 Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$; The vector equation of a line

Exercise 1.1. Give a vector description of the point P that is one-third of the way from A to B on the line segment $\overline{AB}$. Generalize.

Exercise 1.2. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long. (In vector notation, prove that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB}$.)

Exercise 1.3. Prove that the quadrilateral $PQRS$, whose vertices are the midpoints of the sides of an arbitrary quadrilateral $ABCD$, is a parallelogram.

Exercise 1.4. Prove that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

Exercise 1.5. A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. Prove that the three medians of any triangle are **concurrent** (i.e., they have a common point of intersection) at a point G that is two-thirds of the distance from each vertex to the midpoint of the opposite side. (*Hint:* In the triangle ABC, with midpoints PQR, show that the point that is two-thirds of the distance from A to P is given by $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Then show that $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is two-thirds of the distance from B to Q and two-thirds of the distance from C to R.) The point G is called the **centroid** of the triangle.

Exercise 1.6. Let $\mathbf{x} = (1, 2, -2)$ and $\mathbf{y} = (2, -1, 3)$. Determine

(a) $2\mathbf{x} - 3\mathbf{y}$

(b) $-3(\mathbf{x} + 2\mathbf{y}) + 5\mathbf{x}$

(c) $\mathbf{z}$ such that $\mathbf{y} - 2\mathbf{z} = 3\mathbf{x}$

(d) $\mathbf{z}$ such that $\mathbf{z} - 3\mathbf{x} = 2\mathbf{z}$

Exercise 1.7. Consider the points $P(2, 3, 1)$, $Q(3, 1, -2)$, $R(1, 4, 0)$, and $S(-5, 1, 5)$. Determine $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, $\overrightarrow{QR}$, and $\overrightarrow{SR}$, and verify that $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR}$.

Exercise 1.8. Write the vector equation of the line passing through the given point with the given direction vector.

(a) point $(3, 4)$, direction vector $(-5, 1)$

(b) point $(2, 0, 5)$, direction vector $(4, -2, -11)$

(c) point $(4, 0, 1, 5, -3)$, direction vector $(-2, 0, 1, 2, -1)$

Exercise 1.9. Write a vector equation for the line that passes through the given points.

(a) $(-1, 2)$ and $(2, -3)$

(b) $(4, 1)$ and $(-2, -1)$

(c) $(1, 3, -5)$ and $(-2, -1, 0)$

(d) $\left(\frac{1}{2}, \frac{1}{4}, 1\right)$ and $\left(-1, 1, \frac{1}{3}\right)$

(e) $(1, 0, -2, -5)$ and $(-3, 2, -1, 2)$

Exercise 1.10. Find the midpoint of the line segment joining the given points.

(a) $(2, 1, 1)$ and $(-3, 1, -4)$

(b) $(2, -1, 0, 3)$ and $(-3, 2, 1, -1)$

Exercise 1.11. Find points that divide the line segment joining the given points into three equal parts.

(a) $(2, 4, 1)$ and $(-1, 1, 7)$

(b) $(-1, 1, 5)$ and $(4, 2, 1)$

Exercise 1.12. Given the points P and Q, and the real number r, determine the point R such that $\overrightarrow{PR} = r\overrightarrow{PQ}$. Make a rough sketch to illustrate the idea.

(a) $P(1, 4, -5)$ and $Q(-3, 1, 4)$; $r = 1/4$

(b) $P(2, 1, 1, 6)$ and $Q(8, 7, 6, 0)$; $r = -1/3$

(c) $P(2, 1, -2)$ and $Q(-3, 1, 4)$; $r = 4/3$

Exercise 1.13. Determine the points of intersection (if any) of the pairs of lines

(a) $\mathbf{x} = (1, 2) + t(3, 5),\ t \in \mathbb{R}$ and $\mathbf{x} = (3, -1) + s(4, 1),\ s \in \mathbb{R}$

(b) $\mathbf{x} = (2, 3, 4) + t(1, 1, 1),\ t \in \mathbb{R}$ and $\mathbf{x} = (3, 2, 1) + s(3, 1, -1),\ s \in \mathbb{R}$

(c) $\mathbf{x} = (3, 4, 5) + t(1, 1, 1),\ t \in \mathbb{R}$ and $\mathbf{x} = (2, 4, 1) + s(2, 3, -2),\ s \in \mathbb{R}$

(d) $\mathbf{x} = (1, 0, 1) + t(3, -1, 2),\ t \in \mathbb{R}$ and $\mathbf{x} = (5, 0, 7) + s(-2, 2, 2),\ s \in \mathbb{R}$

Exercise 1.14. A set of points in $\mathbb{R}^n$ is **collinear** if they all lie on one line.

(a) By considering directed line segments, give a general method for determining whether a given set of three points is collinear.

(b) Determine whether the points $P(1, 2, 2, 1)$, $Q(4, 1, 4, 2)$, and $R(-5, 4, -2, -1)$ are collinear. Show how you decide.

(c) Determine whether the points $S(1, 0, 1, 2)$, $T(3, -2, 3, 1)$, and $U(-3, 4, -1, 5)$ are collinear. Show how you decide.

Exercise 1.15. Show that $a(5, 7) + b(3, -10) = (-16, 77)$ represents two simultaneous linear equations in the two variables a and b. Solve and check.

Exercise 1.16. Show that $a(1, -1, 0) + b(3, 2, 1) + c(0, 1, 4) = (-1, 1, 19)$ represents three simultaneous linear equations in the three variables a, b, and c. Solve and check.

Answer 1.6. (a) $(-4, 7, -13)$

(b) $(-10, 10 - 22)$

(c) $\mathbf{z} = (-1/2, -7/2, 9/2)$

(d) $\mathbf{z} = (-3, -6, 6)$

Answer 1.7. $\overrightarrow{PQ} = (1, -2, -3)$, $\overrightarrow{PR} = (-1, 1, -1)$, $\overrightarrow{PS} = (-7, -2, 4)$, $\overrightarrow{QR} = (-2, 3, 2)$, $\overrightarrow{SR} = (6, 3, -5)$

Answer 1.8. (a) $\mathbf{x} = (3, 4) + t(-5, 1),\ t \in \mathbb{R}$

(b) $\mathbf{x} = (2, 0, 5) + t(4, -2, -11),\ t \in \mathbb{R}$

(c) $\mathbf{x} = (4, 0, 1, 5, -3) + t(-2, 0, 1, 2, -1),\ t \in \mathbb{R}$

Answer 1.9. Note that alternative correct answers are possible.

(a) $\mathbf{x} = (-1, 2) + t(3, -5),\ t \in \mathbb{R}$

(b) $\mathbf{x} = (4, 1) + t(-6, -2),\ t \in \mathbb{R}$

(c) $\mathbf{x} = (1, 3, -5) + t(-3, -4, 5),\ t \in \mathbb{R}$

(d) $\mathbf{x} = (1/2, 1/4, 1) + t(-3/2, 3/4, -2/3),\ t \in \mathbb{R}$

(e) $\mathbf{x} = (1, 0, -2, -5) + t(-4, 2, 1, 7),\ t \in \mathbb{R}$

Answer 1.10. (a) $(-1/2, 1, -3/2)$ (b) $(-1/2, 1/2, 1/2, 1)$

Answer 1.11. (a) $(1, 3, 3)$ and $(0, 2, 5)$ (b) $(2/3, 4/3, 11/3)$ and $(7/3, 5/3, 7/3)$

Answer 1.12.

(a) $(0, 13/4, -11/4)$ (b) $(0, -1, -2/3, 8)$ (c) $(-14/3, 1, 6)$

Answer 1.13. (a) $(-25/17, -36/17)$

(b) $(0, 1, 2)$

(c) no point of intersection

(d) $(7, -2, 5)$

Answer 1.14. (a) $\overrightarrow{AB} = k\overrightarrow{AC}$ for some $k \in \mathbb{R}$

(b) since $-2\overrightarrow{PQ} = \overrightarrow{PR}$, the points are collinear

(c) S, T, and U are not collinear because $\overrightarrow{SU} \neq k\overrightarrow{ST}$ for any $k \in \mathbb{R}$.

Answer 1.15. We have $\begin{cases} 5a + 3b = -16 \\ 7a - 10b = 77 \end{cases}$ which implies that $a = 1$ and $b = -7$.

Answer 1.16. We have $\begin{cases} a + 3b = -1 \\ -a + 2b + c = 1 \\ b + 4c = 19 \end{cases}$ which implies that $a = 2$, $b = -1$ and $c = 5$.

2 Length; Dot Product; Equation of a Plane

Exercise 2.1. Prove, as a consequence of the triangle inequality, that $|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,| \leq \|\mathbf{x} - \mathbf{y}\|$. (Hint: $\mathbf{x} = \mathbf{x} - \mathbf{y} + \mathbf{y}$)

Exercise 2.2. (a) Let $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$ be non-zero vectors orthogonal to each other, and let $\mathbf{w} = x\mathbf{u_1} + y\mathbf{u_2} + z\mathbf{u_3}$. Show that

$$x = \frac{\mathbf{w} \cdot \mathbf{u_1}}{\|\mathbf{u_1}\|^2}, \quad y = \frac{\mathbf{w} \cdot \mathbf{u_2}}{\|\mathbf{u_2}\|^2}, \quad z = \frac{\mathbf{w} \cdot \mathbf{u_3}}{\|\mathbf{u_3}\|^2}$$

(b) Show that $\mathbf{u_1} = (1, -2, 3)$, $\mathbf{u_2} = (1, 2, 1)$ and $\mathbf{u_3} = (-8, 2, 4)$ are orthogonal to each other. Write $\mathbf{w} = (13, -4, -7)$ as a linear combination of $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$.

Exercise 2.3. An **altitude** of a triangle is a line segment from a vertex that is perpendicular to the opposite side. Prove that the three altitudes of a triangle intersect each other at a common point. (Hint: if A, B, and C are the vertices of the triangle, let H be the point of intersection of the altitudes from A and B. Then prove that $\overrightarrow{CH}$ is orthogonal to $\overrightarrow{AB}$.) (This point is called the **orthocenter** of the triangle.)

Exercise 2.4. Use dot products to prove the Theorem of Thales: If A and B are endpoints of a diameter of a circle, and C is any other point on the circle, then the angle $\angle ABC$ is a right angle. (Hint: Let O be the center of the circle, and express everything in terms of $\mathbf{a}$, $\mathbf{c}$.)

Exercise 2.5. Consider a cube such that each edge has length s. Let the four vertices on one face of the cube be A, B, C, D (taken in order), and A', B', C', D' be the corresponding vertices on the opposite face.

(a) A solid with four vertices (not in a common plane) is called a **tetrahedron**, and is **regular**, if all edges are of equal length. Show that $A'C'BD$ is a regular tetrahedron.

(b) Let P be the centre of the cube. Determine the angle $\angle A'PC'$. (Hint: Locate A at the origin in $\mathbb{R}^3$, let B, D, A' be points on the coordinate axes, and use vectors.)

Exercise 2.6. Let $\mathbf{v}$ be any non-zero vector in $\mathbb{R}^2$, and $\hat{\mathbf{v}}$ be the unit vector in its direction.

(a) Show that $\hat{\mathbf{v}}$ can be written as $\hat{\mathbf{v}} = (\cos\phi, \sin\phi)$, where ϕ is the angle from the positive x-axis to $\mathbf{v}$. Also show that $\mathbf{v} = \|\mathbf{v}\| (\cos\phi, \sin\phi)$.

(b) Prove the formula $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ by considering the dot product of the two unit vectors $\mathbf{e_a} = (\cos\alpha, \sin\alpha)$ and $\mathbf{e_b} = (\cos\beta, \sin\beta)$.

Exercise 2.7. Let $\mathbf{v}$ be any non-zero vector in $\mathbb{R}^3$, and $\hat{\mathbf{v}}$ be the unit vector in its direction. Let α be the angle between $\mathbf{v}$ and the x-axis, let β by the angle between $\mathbf{v}$ and the y-axis, and let γ be the angle between $\mathbf{v}$ and the z-axis.

(a) Show that $\hat{\mathbf{v}} = (\cos\alpha, \cos\beta, \cos\gamma)$, and that $\mathbf{v} = \|\mathbf{v}\| (\cos\alpha, \cos\beta, \cos\gamma)$. (The components of $\hat{\mathbf{v}}$ are called the **direction cosines** of $\mathbf{v}$.)

(b) Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. What familiar formula in $\mathbb{R}^2$ does this correspond to?

(c) Find the direction cosines of $\mathbf{v} = (3, -4, 12)$ and the angles α, β, and γ.

Exercise 2.8. Calculate the lengths of the given vectors.

(a) $(2, -5)$

(b) $(2, 3, -2)$

(c) $\left(1, \frac{1}{5}, -3\right)$

(d) $(1, -1, 0, 2)$

Exercise 2.9. Determine the distance from P to Q if

(a) P is $(2, 3)$ and Q is $(-4, 1)$

(b) P is $(1, 1, -2)$ and Q is $(-3, 1, 1)$

(c) P is $(4, -6, 1)$ and Q is $(-3, 5, 1)$

(d) P is $(2, 1, 1, 5)$ and Q is $(4, 6, -2, 1)$

Exercise 2.10. Verify the triangle inequality and the Cauchy-Schwarz inequality if

(a) $\mathbf{x} = (4, 3, 1)$ and $\mathbf{y} = (2, 1, 5)$

(b) $\mathbf{x} = (1, -1, 2)$ and $\mathbf{y} = (-3, 2, 4)$

Exercise 2.11. Determine the angle (in radians) between the vectors $\mathbf{a}$ and $\mathbf{b}$ if

(a) $\mathbf{a} = (2, 1, 4)$ and $\mathbf{b} = (4, -2, 1)$

(b) $\mathbf{a} = (1, -2, 1)$ and $\mathbf{b} = (3, 1, 0)$

(c) $\mathbf{a} = (5, 1, 1, -2)$ and $\mathbf{b} = (2, 3, -2, 1)$

Exercise 2.12. Determine whether the given pair of vectors if orthogonal.

(a) $(1, 3, 2)$, $(2, -2, 2)$

(b) $(-3, 1, 7)$, $(2, -1, 1)$

(c) $(2, 1, 1)$, $(-1, 4, 2)$

(d) $(4, 1, 0, -2)$, $(-1, 4, 3, 0)$

Exercise 2.13. Determine all values of k for which the vectors are orthogonal.

(a) $(3, -1)$, $(2, k)$

(b) $(3, -1)$, (k, k^2)

(c) $(1, 2, 3)$, $(3, -k, k)$

(d) $(1, 2, 3)$, $(k, k, -k)$

Exercise 2.14. Find the scalar equation of the plane containing the given point with the given normal.

(a) point $(-1, 2, -3)$, normal $(2, 4, -1)$

(b) point $(2, 5, 4)$, normal $(3, 0, 5)$

(c) point $(1, -1, 1)$, normal $(3, -4, 1)$

Exercise 2.15. Determine the scalar equation of the hyperplane passing through the given point with the given normal.

(a) point $(1, 1, -1, -2)$, normal $(3, 1, 4, 1)$

(b) point $(2, -2, 0, 1)$, normal $(0, 1, 3, 3)$

Exercise 2.16. Determine a normal vector for the plane or the hyperplane.

(a) $3x_1 - 2x_2 + x_3 = 7$ in $\mathbb{R}^3$

(b) $-4x_1 + 3x_2 - 5x_3 - 6 = 0$ in $\mathbb{R}^3$

(c) $x_1 - x_2 + 2x_3 - 3x_4 = 5$ in $\mathbb{R}^4$

Exercise 2.17. Find an equation for the plane through the given point and parallel to the given plane.

(a) point $(1, -3, -1)$, plane $2x_1 - 3x_2 + 5x_3 = 17$

(b) point $(0, -2, 4)$, plane $x_2 = 0$

Exercise 2.18. Determine the point of intersection of the given line and plane.

(a) $\mathbf{x} = (2, 3, 1) + t(1, -2, -4)$, $t \in \mathbb{R}$, and $3x_1 - 2x_2 + 5x_3 = 11$

(b) $\mathbf{x} = (1, 1, 2) + t(1, -1, -2)$, $t \in \mathbb{R}$, and $2x_1 + x_2 - x_3 = 5$

Exercise 2.19. Find a vector equation for the plane that passes through the given points.

(a) $(3,2,1)$, $(-4,1,7)$, $(2,0,0)$

(b) $(-1,-4,3)$, $(-2,4,6)$, $(3,1,-4)$

(c) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

Exercise 2.20. Given the plane $2x_1 - x_2 + 3x_3 = 5$, for each of the following lines, determine if the line is parallel to the plane, orthogonal to the plane, or neither parallel nor orthogonal. If the answer is "neither", determine the angle between the direction vector of the line and the normal vector of the plane.

(a) $\mathbf{x} = (3,0,4) + t(-1,1,1)$, $t \in \mathbb{R}$

(b) $\mathbf{x} = (1,1,2) + t(-2,1,-3)$, $t \in \mathbb{R}$

(c) $\mathbf{x} = (3,0,0) + t(1,1,2)$, $t \in \mathbb{R}$

(d) $\mathbf{x} = (-1,-1,2) + t(4,-2,6)$, $t \in \mathbb{R}$

(e) $\mathbf{x} = t(0,3,1)$, $t \in \mathbb{R}$

Exercise 2.21. Determine the equation of the set of points in $\mathbb{R}^3$ that are equidistant from points A and B. Explain why the set is a plane, and determine its normal.

Exercise 2.22. Consider the following statement: "If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ then $\mathbf{b} = \mathbf{c}$."

(a) If the statement is true, prove it. If the statement is false, provide a counterexample. (A counterexample is an example, meaning specific vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, for which the statement is false.)

(b) If we specify $\mathbf{a} \neq \mathbf{0}$, does that change the result?

Exercise 2.23. Prove the parallelogram law for the norm:

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\,\|\mathbf{a}\|^2 + 2\,\|\mathbf{b}\|^2$$

for all vectors in $\mathbb{R}^n$. Interpret geometrically!

Answer 2.1. From the hint:

$$\begin{aligned} \|\mathbf{x}\| &= \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| \\ &\leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\| \quad \text{by the triangle inequality} \end{aligned}$$

Rearranging, we get $\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|$.

Starting again with the identity $-\mathbf{y} = \mathbf{x} - \mathbf{y} + (-\mathbf{x})$, and using the fact that $\|-\mathbf{y}\| = \|\mathbf{y}\|$ we get

$$\|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}\|\,,$$

which gives that $\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\|$.

These two inequalities together give the desired result.

Answer 2.2. (a) Take the equation $\mathbf{w} = x\mathbf{u_1} + y\mathbf{u_2} + z\mathbf{u_3}$, and dot the whole thing with $\mathbf{u_1}$. This gives

$$\mathbf{w}\cdot\mathbf{u_1} = x(\mathbf{u_1}\cdot\mathbf{u_1}) + y(\mathbf{u_2}\cdot\mathbf{u_1}) + z(\mathbf{u_3}\cdot\mathbf{u_1}).$$

Since $\mathbf{u_1}\cdot\mathbf{u_1} = \|\mathbf{u_1}\|^2$ and $\mathbf{u_2}\cdot\mathbf{u_1} = \mathbf{u_3}\cdot\mathbf{u_1} = 0$ (why?), we can solve for x to get

$$x = \frac{\mathbf{w}\cdot\mathbf{u_1}}{\|\mathbf{u_1}\|^2}.$$

Taking the original equation and dotting with $\mathbf{u_2}$ and $\mathbf{u_3}$ will give the formulas for y and z, respectively.

(b) First calculate all the dot products:

$$\begin{aligned}
\mathbf{u_1} \cdot \mathbf{u_2} &= (1, -2, 3) \cdot (1, 2, 1) = 1 - 4 + 3 = 0 \checkmark \\
\mathbf{u_1} \cdot \mathbf{u_3} &= (1, -2, 3) \cdot (-8, 2, 4) = -8 - 4 + 12 = 0 \checkmark \\
\mathbf{u_2} \cdot \mathbf{u_3} &= (1, 2, 1) \cdot (-8, 2, 4) = -8 + 4 + 4 = 0 \checkmark
\end{aligned}$$

Now use the formulas from part (a) to get x, y, and z, and see that

$$\begin{aligned}
\mathbf{w} &= 0\mathbf{u_1} - \frac{2}{6}\mathbf{u_2} - \frac{140}{84}\mathbf{u_3} \\
&= -\frac{1}{3}(1, 2, 1) - \frac{5}{3}(-8, 2, 4) \\
&= \left(-\frac{1}{3} + \frac{40}{3}, -\frac{2}{3} - \frac{10}{3}, -\frac{1}{3} - \frac{20}{3}\right) \\
&- (13, -4, -7) = \mathbf{w} \checkmark
\end{aligned}$$

Answer 2.3. Assume that $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$ and $\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$. Therefore,

$$\begin{aligned}
\overrightarrow{AH} \cdot \overrightarrow{BC} &= \overrightarrow{BH} \cdot \overrightarrow{AC} \\
(\mathbf{h} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) &= (\mathbf{h} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) \\
\mathbf{h} \cdot \mathbf{c} - \mathbf{h} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{b} &= \mathbf{h} \cdot \mathbf{c} - \mathbf{h} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a}
\end{aligned}$$

Here we can cancel out $\mathbf{h} \cdot \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b}$ from both sides, leaving us with

$$\begin{aligned}
-\mathbf{h} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} &= -\mathbf{h} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} \\
0 &= \mathbf{h} \cdot \mathbf{b} - \mathbf{h} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{a} \\
0 &= (\mathbf{h} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) \\
0 &= \overrightarrow{CH} \cdot \overrightarrow{AB}
\end{aligned}$$

and we are done.

Answer 2.4. We know that $\mathbf{b} = -\mathbf{a}$, and that $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\|$. We want to show that $\overrightarrow{AC} \cdot \overrightarrow{CB} = 0$. Indeed,

$$\begin{aligned}
\overrightarrow{AC} \cdot \overrightarrow{CB} &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) \\
&= \mathbf{c} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \\
&= \mathbf{c} \cdot (-\mathbf{a}) - \|\mathbf{c}\|^2 - \mathbf{a} \cdot (-\mathbf{a}) + \mathbf{a} \cdot \mathbf{c} \\
&= -\mathbf{a} \cdot \mathbf{c} - \|\mathbf{c}\|^2 + \|\mathbf{a}\|^2 + \mathbf{a} \cdot \mathbf{c} \\
&= 0
\end{aligned}$$

Answer 2.5. (a) Each of $\overrightarrow{A'C'}$, $\overrightarrow{A'B}$, $\overrightarrow{A'D}$, $\overrightarrow{C'B}$, $\overrightarrow{C'D}$, and $\overrightarrow{BD}$ is the diagonal of a square of side length s, so each has norm $= \sqrt{2}s$, so the tetrahedron is regular.

(b) Suppose $A(0, 0, 0)$, $A'(s, 0, 0)$, $B(0, s, 0)$, and $D(0, 0, s)$. Then $P(s/2, s/2, s/2)$ and $C'(s, s, s)$. This gives that $\overrightarrow{PA'} = (s, 0, 0) - (s/2, s/2, s/2) = (s/2, -s/2, -s/2)$ and $\overrightarrow{PC'} = (s, s, s) -$

$(s/2, s/2, s/2) = (s/2, s/2, s/2)$. Now use the formula for the cosine of the angle:

$$\begin{aligned}\cos\theta &= \frac{\overrightarrow{PA'} \cdot \overrightarrow{PC'}}{\left\|\overrightarrow{PA'}\right\| \left\|\overrightarrow{PC'}\right\|} \\ &= \frac{-s^2/4}{((\sqrt{3}/2)s)^2} \\ &= -\frac{1}{3}\end{aligned}$$

which gives $\theta \approx 1.91$ radians $\approx 109.47°$.

Answer 2.8. (a) $\sqrt{29}$ (b) $\sqrt{17}$ (c) $\sqrt{251}/5$ (d) $\sqrt{6}$

Answer 2.9. (a) $2\sqrt{10}$ (b) 5 (c) $\sqrt{170}$ (d) $3\sqrt{6}$

Answer 2.11. (a) ≈ 1.074 radians (c) ≈ 1.180 radians

(b) ≈ 1.441 radians

Answer 2.12. (a) Yes (c) No

(b) Yes (d) Yes

Answer 2.13. (a) $k = 6$ (c) $k = -3$

(b) $k = 0$ or $k = 3$ (d) any $k \in \mathbb{R}$

Answer 2.14. (a) $2x_1 + 4x_2 - x_3 = 9$ (c) $3x_1 - 4x_2 + x_3 = 8$

(b) $3x_1 + 5x_3 = 26$

Answer 2.15. (a) $3x_1 + x_2 + 4x_3 + x_4 = -2$ (b) $x_2 + 3x_3 + 3x_4 = 1$

Answer 2.16. (a) $\mathbf{n} = (3, -2, 1)$ (c) $\mathbf{n} = (1, -1, 2, -3)$

(b) $\mathbf{n} = (-4, 3, -5)$

Answer 2.17. (a) $2x_1 - 3x_2 + 5x_3 = 6$ (b) $x_2 = -2$

Answer 2.18. (a) $(20/13, 51/13, 37/13)$ (b) $(7/3, -1/3, -2/3)$

Answer 2.19. (a) $\mathbf{x} = (3, 2, 1) + r(-7, -1,) + s(-1, -2, 1),\ r, s \in \mathbb{R}$

(b) $\mathbf{x} = (-1, -4, 3) + r(-1, 8, 3) + s(4, 5, -7),\ r, s \in \mathbb{R}$

(c) $\mathbf{x} = (1, 0, 0) + r(-1, 1, 0) + s(-1, 0, 1),\ r, s \in \mathbb{R}$

Answer 2.20. (a) The line is parallel to the plane.

(b) The line is orthogonal to the plane.

(c) The line is neither parallel nor orthogonal to the plane, $\theta \approx 0.702$ radians.

(d) The line is orthogonal to the plane.

(e) The line is parallel to the plane.

3 Projection and Minimum Distance

Exercise 3.1. For each of the given pairs of vectors $\mathbf{a}$, $\mathbf{b}$, check that $\mathbf{a}$ is a unit vector, determine $\text{proj}_{\mathbf{a}}(\mathbf{b})$ and $\text{perp}_{\mathbf{a}}(\mathbf{b})$, and check your results by verifying that $\text{proj}_{\mathbf{a}}(\mathbf{b}) + \text{perp}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b}$ and $\mathbf{a} \cdot \text{perp}_{\mathbf{a}}(\mathbf{b}) = 0$ in each case.

(a) $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (3, -5)$

(b) $\mathbf{a} = (3/5, 4/5)$ and $\mathbf{b} = (-4, 6)$

(c) $\mathbf{a} = (0, 1, 0)$ and $\mathbf{b} = (-3, 5, 2)$

(d) $\mathbf{a} = (1/3, -2/3, 2/3)$ and $\mathbf{b} = (4, 1, -3)$

Exercise 3.2. Consider the force represented by the vector $\mathbf{F} = (10, 18, -6)$, and let $\mathbf{u} = (2, 6, 3)$.

(a) Determine a unit vector in the direction of $\mathbf{u}$.

(b) Determine the projection of $\mathbf{F}$ onto $\mathbf{u}$.

(c) Determine the projection of $\mathbf{F}$ perpendicular to $\mathbf{u}$.

Exercise 3.3. The same instructions exercise 3.2 with $\mathbf{F} = (3, 11, 2)$ and $\mathbf{u} = (3, 1, -2)$.

Exercise 3.4. Determine

(a) $\text{proj}_{(2,3,-2)}(4, -1, 3)$ and $\text{perp}_{(2,3,-2)}(4, -1, 3)$

(b) $\text{proj}_{(1,1,-2)}(4, 1, -2)$ and $\text{perp}_{(1,1,-2)}(4, 1, -2)$

(c) $\text{proj}_{(-2,1,-1)}(5, -1, 3)$ and $\text{perp}_{(-2,1,-1)}(5, -1, 3)$

(d) $\text{proj}_{(-1,2,1,-3)}(2, -1, 2, 1)$ and $\text{perp}_{(-1,2,1,-3)}(2, -1, 2, 1)$

Exercise 3.5. For the given point and line, find by projection the point on the line that is closest to the given point, and use perp to find the distance from the point to the line.

(a) point $(0, 0)$, line $\mathbf{x} = (1, 4) + t(-2, 2),\ t \in \mathbb{R}$

(b) point $(2, 5)$, line $\mathbf{x} = (3, 7) + t(1, -4),\ t \in \mathbb{R}$

(c) point $(1, 0, 1)$, line $\mathbf{x} = (2, 2, -1) + t(1, -2, 1),\ t \in \mathbb{R}$

(d) point $(2, 3, 2)$, line $\mathbf{x} = (1, 1, -1) + t(1, 4, 1),\ t \in \mathbb{R}$

Exercise 3.6. Use a projection (onto or perpendicular to) to find the distance from the point to the plane.

(a) point $(2, 3, 1)$, plane $3x_1 - x_2 + 4x_3 = 5$

(b) point $(-2, 3, -1)$, plane $2x_1 - 3x_2 - 5x_3 = 5$

(c) point $(0, 2, -1)$, plane $2x_1 - x_3 = 5$

(d) point $(-1,-1,1)$, plane $2x_1 - x_2 - x_3 = 4$

Exercise 3.7. For the given point and hyperplane in $\mathbb{R}^4$, determine by a projection the point in the hyperplane that is closest to the given point.

(a) point $(2,4,3,4)$, hyperplane $3x_1 - x_2 + 4x_3 + x_4 = 0$

(b) point $(-1,3,2,-1)$, hyperplane $x_1 + 2x_2 + x_3 - x_4 = 4$

Answer 3.1. (a) $\text{proj}_{\mathbf{a}}\mathbf{b} = (0,-5)$, $\text{perp}_{\mathbf{a}}\mathbf{b} = (3,0)$

(b) $\text{proj}_{\mathbf{a}}\mathbf{b} = (36/25, 48/25)$, $\text{perp}_{\mathbf{a}}\mathbf{b} = (-136/25, 102/25)$

(c) $\text{proj}_{\mathbf{a}}\mathbf{b} = (0,5,0)$, $\text{perp}_{\mathbf{a}}\mathbf{b} = (-3,0,2)$

(d) $\text{proj}_{\mathbf{a}}\mathbf{b} = (-4/9, 8/9, -8/9)$, $\text{perp}_{\mathbf{a}}\mathbf{b} = (40/9, 1/9, -19/9)$

Answer 3.2. (a) $\hat{\mathbf{u}} = (2/7, 6/7, 3/7)$ (c) $(270/49, 222/49, -624/49)$

(b) $(220/49, 660/49, 330/49)$

Answer 3.3. (a) $\hat{\mathbf{u}} = (3/\sqrt{14}, 1/\sqrt{14}, -2/\sqrt{14})$ (c) $(-3/7, 69/7, 30/7)$

(b) $(24/7, 8/7, -16/7)$

Answer 3.4.

(a) $(-2/17, -3/17, 2/17)$, $(70/17, -14/17, 49/17)$ (c) $(14/3, -7/3, 7/3)$, $(1/3, 4/3, 2/3)$

(b) $(3/2, 3/2, -3)$, $(5/2, -1/2, 1)$ (d) $(1/3, -2/3, -1/3, 1)$, $(5/3, -1/3, 7/3, 0)$

Answer 3.5. (a) $(5/2, 5/2)$, $5/\sqrt{2}$ (c) $(17/6, 1/3, -1/6)$, $\sqrt{29/6}$

(b) $(58/17, 91/17)$, $6/\sqrt{17}$ (d) $(5/3, 11/3, -1/3)$, $\sqrt{6}$

Answer 3.6. (a) $\dfrac{2}{\sqrt{26}}$ (c) $\dfrac{4}{\sqrt{5}}$

(b) $\dfrac{13}{\sqrt{38}}$ (d) $\sqrt{6}$

Answer 3.7. (a) $\frac{1}{3}(0, 14, 1, 10)$ (b) $\frac{1}{7}(-11, 13, 10, -3)$

4 Cross Products and Volumes

Exercise 4.1. Calculate the following cross products.

(a) $(1,-5,2) \times (-2,1,5)$ (b) $(2,-3,-5) \times (4,-2,7)$ (c) $(-1,0,1) \times (0,4,5)$

Exercise 4.2. Let $\mathbf{p} = (-1,4,2)$, $\mathbf{q} = (3,1,-1)$, and $\mathbf{r} = (2,-3,-1)$. Check by calculation that the following general properties hold.

(a) $\mathbf{p} \times \mathbf{p} = \mathbf{0}$

(b) $\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}$

(c) $\mathbf{p} \times 3\mathbf{r} = 3(\mathbf{p} \times \mathbf{r})$

(d) $\mathbf{p} \times (\mathbf{q} + \mathbf{r}) = \mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{r}$

(e) $\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) \neq (\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$

Exercise 4.3. Calculate the area of the parallelogram determined by the following vectors. (Hint: For the vectors in $\mathbb{R}^2$, think of them as vectors in $\mathbb{R}^3$ by letting $z = 0$.)

(a) $(1, 2, 1)$ and $(2, 3, -1)$

(b) $(1, 0, 1)$ and $(1, 1, 4)$

(c) $(1, 2)$ and $(-2, 5)$

(d) $(-3, 1)$ and $(4, 3)$

Exercise 4.4. In each case, determine whether the given pair of lines has a point of intersection; if so, determine the scalar equation of the plane containing the lines, and if not, determine the distance between the lines.

(a) $\mathbf{x} = (1, 3, 1) + s(-2, -1, 1)$ and $\mathbf{x} = (0, 1, 4) + t(3, 0, 1)$, $s, t \in \mathbb{R}$

(b) $\mathbf{x} = (1, 3, 1) + s(-2, -1, 1)$ and $\mathbf{x} = (0, 1, 7) + t(3, 0, 1)$, $s, t \in \mathbb{R}$

(c) $\mathbf{x} = (2, 1, 4) + s(2, 1, -2)$ and $\mathbf{x} = (-2, 1, 5) + t(1, 3, 1)$, $s, t \in \mathbb{R}$

(d) $\mathbf{x} = (0, 1, 3) + s(1, -1, 4)$ and $\mathbf{x} = (0, -1, 5) + t(1, 1, 2)$, $s, t \in \mathbb{R}$

Exercise 4.5. Determine the scalar equation of the plane with the given vector equation.

(a) $\mathbf{x} = (1, 4, 7) + s(2, 3, -1) + t(4, 1, 0)$, $s, t \in \mathbb{R}$

(b) $\mathbf{x} = (2, 3, -1) + s(1, 1, 0) + t(-2, 1, 2)$, $s, t \in \mathbb{R}$

(c) $\mathbf{x} = (1, -1, 3) + s(2, -2, 1) + t(0, 3, 1)$, $s, t \in \mathbb{R}$

Exercise 4.6. Determine the scalar equation of the plane that contains the following points.

(a) $(2, 1, 5), (4, -3, 2), (2, 6, -1)$

(b) $(3, 1, 4), (-2, 0, 2), (1, 4, -1)$

(c) $(-1, 4, 2), (3, 1, -1), (2, -3, -1)$

Exercise 4.7. Determine a vector equation of the line of intersection of the given planes.

(a) $x + 3y - z = 5$ and $2x - 5y + z = 7$

(b) $2x - 3z = 7$ and $y + 2z = 4$

Exercise 4.8. Find the volume of the parallelepiped determined by the following vectors.

(a) $(4, 1, -1)$, $(-1, 5, 2)$, and $(1, 1, 6)$

(b) $(-2, 1, 2)$, $(3, 1, 2)$, and $(0, 2, 5)$

Answer 4.1. (a) $(-27, -9, -9)$ (b) $(-31, -34, 8)$ (c) $(-4, 5, -4)$

Answer 4.3. (a) $\sqrt{35}$ (b) $\sqrt{11}$ (c) 9 (d) 13

Answer 4.4. (a) Point of intersection: $(-3, 1, 3)$, $-x + 5y + 3z = 17$

(b) No point of intersection, $9/\sqrt{35}$

(c) No point of intersection, $23/3\sqrt{10}$

(d) Point of intersection $(1, 0, 7)$, $3x - y - z = -4$

Answer 4.5. (a) $x - 4y - 10z = -85$ (c) $-5x - 2y + 6z = 15$

(b) $2x - 2y + 3z = -5$

Answer 4.6. (a) $39x + 12y + 10z = 140$ (c) $-12x + 3y - 19z = -14$

(b) $11x - 21y - 17z = -56$

Answer 4.7. (a) $\mathbf{x} = (46/11, 3/11, 0) + t(-2, -3, -11),\ t \in \mathbb{R}$

(b) $\mathbf{x} = (7/2, 4, 0) + t(3, -4, 2),\ t \in \mathbb{R}$

Answer 4.8. (a) 126 (b) 5

5 Exercises on Lines and Planes

Exercise 5.1. Find the equations (in vector form and parametric form) of the line passing through the given point and parallel to the given vector.

(a) $A(1, -1, -1)$, $\mathbf{d} = (2, 3, -1)$ (b) $B(2, -4, 5)$, $\mathbf{d} = 3\hat{\mathbf{\imath}} - \hat{\mathbf{\jmath}} - 2\hat{\mathbf{k}}$

Exercise 5.2. Find the equations of the lines through $P(3, -4, 7)$ which are parallel to the coordinate axes.

Exercise 5.3. Find the parametric equations of the line through the given point and parallel to the line with given equations

(a) $A(0, 0, 2)$, $\mathbf{x} = (1, 2, -1) + t(2, 3, -3)$

(b) $B(1, 0, 0)$, $x = -1 + 2t,\ y = -1 + 3t,\ z = 1 - 2t$

Exercise 5.4. Find the parametric equations of the line through the given point and orthogonal to the plane with given equations.

(a) $A(-5, 1, 6)$, $2x - 6y + 3z = 9$ (b) $B(1, 0, 2)$, $x - 2y = 1$

Exercise 5.5. Find the equations of the line through $A(1, 3, -2)$ and parallel to the planes with equations $2x - y + z = 2$ and $x + y - 2z = 5$.

Exercise 5.6. Find the equations of the line through $P(1, 0, 3)$ that is orthogonal to the lines $\mathbf{x} = (5, 5, -2) + s(1, 2, -3)$ and $\mathbf{x} = (2, 0, 1) + t(1, 1, -2)$.

Exercise 5.7. Find the equations of the line containing $A(-1, 3, -2)$ and $B(3, 2, 1)$.

Exercise 5.8. Find the angle between the direction vectors of the lines with equations $\mathbf{x} = (0, 1, -1) + s(4, 0, -3)$ and $\mathbf{x} = (2, -3, 1) + t(2, -1, 2)$.

Exercise 5.9. Find the equation of the plane through the given point and orthogonal to the given vector.

(a) $A(1,0,-1)$, $\mathbf{n} = (3,2,-1)$

(b) $A(2,1,3)$, $\mathbf{n} = (5,4,7)$

(c) $A(0,0,0)$, $\mathbf{n} = (1,-1,0)$

(d) $A(0,1,1)$, $\mathbf{n} = (1,0,0)$

Exercise 5.10. Given the two points $A(-1,2,-3)$ and $B(-1,1,3)$, find the equation of the plane through A which is orthogonal to $\overrightarrow{AB}$.

Exercise 5.11. Find the equation of the plane which bisects the line segment joining $A(3,5,-7)$ to $B(3,3,3)$ at a right angle.

Exercise 5.12. Find the equations of the three coordinate planes.

Exercise 5.13. Find the equation of the plane which contains the three points.

(a) $A(1,2,1)$, $B(2,3,1)$, $C(3,-1,2)$

(b) $A(1,1,0)$, $B(0,1,2)$, $C(3,0,2)$

Exercise 5.14. Find the equation of the plane through $A(1,3,-2)$ which is parallel to the plane $3x + y + 4z = 3$.

Exercise 5.15. Find the equation of the plane which contains the given point and is orthogonal to the two given planes.

(a) $P(1,1,1)$, $x + y + z = 3$ and $2x + y - z = 4$

(b) $P(2,1,-2)$, $2x + 3z = 1$ and $y = 3$

Exercise 5.16. Find the equation of the plane passing through $A(2,0,5)$ and $B(0,2,-1)$ and orthogonal to the plane $x + 3y - z = 7$.

Exercise 5.17. Given the plane $2x - 3y + 4z + 10 = 0$, determine whether the given line is parallel to the plane. If the plane is not parallel, find the point of intersection of the line and the plane. If the line is parallel, determine whether it is outside the plane, or lies entirely within it.

(a) $x = 2 - 3t,\ y = 6 + 2t,\ z = 1 + 3t$

(b) $x = 2 - t,\ y = 1 + t,\ z = 1 + 2t$

(c) $x = 1 + t,\ y = 1 + 2t,\ z = 1 + t$

Exercise 5.18. For the following pairs of lines, do the following:

(i) If the lines are parallel, find the distance between them.

(ii) If the lines are not parallel, determine whether the lines intersect, and if so, find the point of intersection.

(iii) If the lines are skew, find the distance between them.

(a) $\mathbf{x} = (-1,-1,-4) + t(1,2,1)$ and $\mathbf{x} = (1,4,4) + s(-1,3,-1)$

(b) $\mathbf{x} = (2,-3,-3) + t(1,2,4)$ and $\mathbf{x} = (3,2,4) + s(1,-1,1)$

(c) $\mathbf{x} = (0,4,5) + t(2,-1,1)$ and $\mathbf{x} = (1,2,1) + s(-4,2,-2)$

Exercise 5.19. For each pair of lines, determine whether the lines intersect each other. If so, determine the point of intersection.

(a) $\mathbf{x} = (2,-1,3) + t(-3,4,3)$ and $\mathbf{x} = (-6,9,25) + s(2,-3,5)$

(b) $\mathbf{x} = (-1, 2, 0) + t(5, -8, 4)$ and $\mathbf{x} = (18, -18, 6) + s(-3, 10, -7)$

(c) $\mathbf{x} = (0, 4, 2) + t(-1, 1, 1)$ and $\mathbf{x} = (-4, -2, 9) + s(1, 1, -2)$

(d) $\mathbf{x} = (-9, 6, 2) + t(-3, -4, 6)$ and $\mathbf{x} = (-34, -21, 40) + s(7, 3, -2)$

(e) $\mathbf{x} = (4, 3, 0) + t(-1, 5, 6)$ and $\mathbf{x} = (-3, 5, 4) + s(1, 2, -2)$

Exercise 5.20. For each pair of skew lines, find the minimum distance between them.

(a) $\mathbf{x} = (0, 4, 2) + t(-1, 1, 1)$ and $\mathbf{x} = (-4, -2, 9) + s(1, 1, -2)$

(b) $\mathbf{x} = (6, 1, 3) + t(2, -1, 3)$ and $\mathbf{x} = (9, 2, -3) + s(3, 1, 5)$

(c) $\mathbf{x} = (5, 0, -2) + t(3, -2, 4)$ and $\mathbf{x} = (1, 2, -5) + s(6, 1, 7)$

(d) $\mathbf{x} = (3, 9, -1) + t(0, -1, 3)$ and $\mathbf{x} = (5, -3, 0) + s(2, 3, -1)$

Exercise 5.21. Find the point of the plane which is closest to the point P.

(a) $5x - y + z = 14$ and $P(-2, 3, 0)$

(b) $x + 2y - 3z = 25$ and $P(4, 1, 3)$

Exercise 5.22. Find the point on the line which is closest to the point P.

(a) $x = 2 - t,\ y = 3 + 2t,\ z = -2 + t$, and $P(5, 6, 1)$

(b) $x = 4 - 2t,\ y = -2 + 3t,\ z = 2t$, and $P(1, 5, 12)$

Exercise 5.23. Given:

Points : $A(1, 1, 2), B(2, -1, 0), C(-1, 1, 0)$

Planes : $\mathcal{P}_1 : 2x + 2y - 4z = 5$
$\mathcal{P}_2 : x + 2y + z = 1$
$\mathcal{P}_3 : x + y - 2z = 0$

Lines : $\mathcal{L}_1 : \mathbf{x} = (2, 1, 3) + t(4, 2, -3),\ t \in \mathbb{R}$
$\mathcal{L}_2 : \mathbf{x} = (3, 1, 2) + t(1, -1, -2),\ t \in \mathbb{R}$
$\mathcal{L}_3 : \mathbf{x} = (3, -1, 4) + t(-2, 2, 4),\ t \in \mathbb{R}$

(a) Find the distance between each pair of points. (There are three distances to be found.)

(b) Find the distance from each point to each plane. (There are nine distances to be found.)

(c) Find the distance from each point to each line. (There are nine distances to be found.)

(d) Find the distance between each pair of lines. (There are three distances to be found.)

(e) Find the distance between $\mathcal{P}_1$ and $\mathcal{P}_3$, (the only pair of parallel planes).

(f) Find the point on each line which is closest to each point. (There are nine points to be found.)

(g) Find the point in each plane which is closest to each point. (There are nine points to be found.)

(h) For each pair of lines find the point on each which is closest to the other. (This is not possible for one pair on lines, explain why not.)

Exercise 5.24. Find equations for the line of intersection of the given planes.

(a) $7x - 2y + 3z = -2$ and $-3x + y + 2z + 5 = 0$

(b) $2x + 3y - 5z = 0$ and $y = 0$

Exercise 5.25. Find the point of intersection of the line $\mathbf{x} = (9, -1, 3) + t(-5, -1, 1)$ and the plane $2x - 3y + 4z + 7 = 0$.

Exercise 5.26. Find an equation for the plane that contains the line $x = -1 + 3t,\ y = 5 + 2t,\ z = 2 - t$ and is orthogonal to the plane $2x - 4y + 2z = 9$.

Exercise 5.27. Find an equation for the plane that passes through $(2, 4, -1)$ and contains the line of intersection of the planes $x - y - 4z = 2$ and $-2x + y + 2z = 3$.

Exercise 5.28. Show that the points $(-1, -2, -3)$, $(-2, 0, 1)$, $(-4, -1, -1)$, and $(2, 0, 1)$ lie in the same plane.

Exercise 5.29. Find an equation for the plane through $(2, -1, 4)$ that is orthogonal to the line of intersection of the planes $4x + 2y + 2z = -1$ and $3x + 6y + 3z = 7$.

Exercise 5.30. Find an equation for the plane that is orthogonal to the plane $8x - 2y + 6z = 1$ and passes through the points $P(-1, 2, 5)$ and $Q(2, 1, 4)$.

Exercise 5.31. Find an equation for the plane that contains the point $(1, -1, 2)$ and the line $x = t,\ y = 1 + t,\ z = -3 + 2t$.

Exercise 5.32. Find an equation for the plane that contains the line $x = 1 + t,\ y = 3t,\ z = 2t$ and is parallel to the line of intersection of the planes $-x + 2y + z = 0$ and $x + z + 1 = 0$.

Exercise 5.33. Find an equation for the plane, each of whose points are equidistant from $(-1, -4, -2)$, and $(0, -2, 2)$.

Exercise 5.34. Two intersecting planes in $\mathbb{R}^3$ determine two angles of intersection: an acute angle $(0 \le \theta \le \pi/2)$ and its supplement $\pi - \theta$. If $\mathbf{n_1}$ and $\mathbf{n_2}$ are non-zero normals to the planes then the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ is θ or $\pi - \theta$, depending on the directions of the normals. For each given pair of planes, find the acute angle of intersection of the planes.

(a) $x = 0$ and $2x - y + z - 4 = 0$

(b) $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

Answer 5.1. (a) $\mathbf{x} = (1, -1, -1) + t(2, 3, -1)$, $x = 1 + 2t,\ y = -1 + 3t,\ z = -1 - t$

Answer 5.3. (a) $\mathbf{x} = (0, 0, 2) + t(2, 3, -3)$

Answer 5.5. $\mathbf{x} = (1, 3, -2) + t(1, 5, 3)$

Answer 5.7. $\mathbf{x} = (-1, 3, -2) + t(4, -1, 3)$

Answer 5.9. (a) $3x + 2y - z = 4$ (c) $x - y = 0$

Answer 5.11. $-y + 5z = -14$

Answer 5.13. (a) $x - y - 5z = -6$

Answer 5.15. (a) $-2x + 3y - z = 0$

Answer 5.17. (a) parallel and in the plane

Answer 5.19. (a) $(-10, 15, 15)$

(c) no point of intersection

Answer 5.21. (a) $(3, 2, 1)$

Answer 5.23. (a) $\mathrm{d}(A, B) = 3$

(b) $\mathrm{d}(A, \mathcal{P}_1) = \frac{9}{2\sqrt{6}}$

(c) $\mathrm{d}(A, \mathcal{L}_1) = \frac{\sqrt{57}}{\sqrt{29}}$

(d) $\mathrm{d}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{\sqrt{110}}$

(e) $\frac{5}{2\sqrt{6}}$

(f) $\mathrm{pt}(A, \mathcal{L}_1) = \frac{3}{29}(18, 9, 30)$

(g) $\mathrm{pt}(A, \mathcal{P}_1) = \frac{1}{4}(7, 7, 2)$

(h) $\mathrm{pts}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{55}(146, 73, 138), \frac{1}{110}(299, 141, 282)$

Answer 5.25. $\frac{1}{3}(-173, -43, 49)$

Answer 5.27. $x - y - 4z = 2$

Answer 5.29. $x + y - 3z = -11$

Answer 5.31. $3x - y - z = 2$

Answer 5.33. $2x + 4y + 8z = -13$

6 Review Questions

Exercise 6.1. Find the point where the line through $A(3, -2, 7)$ and $B(13, 3, -8)$ meets the xz-plane.

Exercise 6.2. Prove that the point $(2, 1, 4)$, $(1, -1, 2)$, $(3, 3, 6)$ are collinear.

Exercise 6.3. If $A(2, 3, -1)$ and $B(3, 7, 4)$, find the points P on the line through A and B satisfying $\left\|\overrightarrow{PA}\right\| / \left\|\overrightarrow{PB}\right\| = 2/5$.

Exercise 6.4. Let $\mathcal{M}$ be the line through $A(1, 2, 3)$ parallel to the line joining $B(-2, 2, 0)$ and $C(4, -1, 7)$. Also let $\mathcal{N}$ be the line joining $E(1, -1, 8)$ and $F(10, -1, 11)$ Prove that $\mathcal{M}$ and $\mathcal{N}$ intersect and find the point of intersection.

Exercise 6.5. Prove that the triangle formed by the points $(-3, 5, 6)$, $(-2, 7, 9)$ and $(2, 1, 7)$ is a $30°$, $60°$, $90°$ triangle.

Exercise 6.6. Find the point on the line through $A(-2, 1, 3)$ and $B(1, 2, 4)$ which is closest to the origin. Also find this shortest distance.

Exercise 6.7. A line $\mathcal{N}$ is determined by the two planes $x + y - 2z = 1$ and $x + 3y - z = 4$. Find the point P on $\mathcal{N}$ closest to the point $C(1, 0, 1)$ and find the distance from P to C.

Exercise 6.8. Find a scalar equation describing the plane perpendicular to the line of intersection of the planes $x + y - 2z = 4$ and $3x - 2y + z = 1$ and which passes through $(6, 0, 2)$.

Exercise 6.9. Find the length of the projection of the vector $\overrightarrow{AB}$ onto the line $\mathcal{L}$, where $A(1, 2, 3)$, $B(5, -2, 6)$, and $\mathcal{L}$ is the line through $C(7, 1, 9)$ and $D(-1, 5, 8)$

Exercise 6.10. Find a scalar equation for the plane through $A(3, -1, 2)$, perpendicular to the line $\mathcal{L}$ joining $B(2, 1, 4)$ and $C(-3, -1, 7)$. Also find the point of intersection of $\mathcal{L}$ and the plane and hence determine the distance from A to $\mathcal{L}$.

Exercise 6.11. If P is a point inside the triangle ABC, prove that

$$\mathbf{p} = r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

where $r + s + t = 1$ and $r > 0,\ s > 0,\ t > 0$.

Exercise 6.12. If B is the point where the perpendicular from $A(6, -1, 11)$ meets the plane $3x + 4y + 5z = 10$, find B and the distance from A to B.

Exercise 6.13. Prove that the triangle with vertices $(-3, 0, 2)$, $(6, 1, 4)$, and $(-5, 1, 0)$ has area $\frac{1}{2}\sqrt{333}$.

Exercise 6.14. Find an equation for the plane through $(2, 1, 4)$, $(1, -1, 2)$, and $(4, -1, 1)$.

Exercise 6.15. Lines $\mathcal{L}$ and $\mathcal{M}$ are non-parallel in $\mathbb{R}^3$ and are given by equations

$$\mathbf{p} = \mathbf{a} + s\mathbf{c}, \qquad \mathbf{q} = \mathbf{b} + t\mathbf{d}.$$

(a) Prove that there is precisely one pair of points P and Q such that $\overrightarrow{PQ}$ is perpendicular to $\mathbf{c}$ and $\mathbf{d}$.

(b) Explain why $\left\|\overrightarrow{PQ}\right\|$ is the shortest distance between lines $\mathcal{L}$ and $\mathcal{M}$. Also prove that

$$\left\|\overrightarrow{PQ}\right\| = \frac{|(\mathbf{c} \times \mathbf{d}) \cdot \overrightarrow{AB}|}{\|\mathbf{c} \times \mathbf{d}\|}.$$

Exercise 6.16. The points $A(1, 1, 5)$, $B(2, 2, 1)$, $C(1, -2, 2)$ and $D(-2, 1, 2)$ are the vertices of a tetrahedron. Find the equation of the line through A perpendicular to the face BCD and the distance of A from this face. Also find the shortest distance between the skew lines $\mathcal{M}$ and $\mathcal{N}$, where $\mathcal{M}$ goes through A and D, and $\mathcal{N}$ goes through B and C.

Answer 6.1. $(7, 0, 1)$

Answer 6.3. $(16/7, 29/7, 3/7)$ and $(4/3, 1/3, -13/3)$

Answer 6.4. $(7, -1, 10)$

Answer 6.6. $(-16/11, 13/11, 35/11)$ and $\sqrt{150/11}$

Answer 6.7. $(4/3, 17/15, 11/15)$ and $\sqrt{330}/15$

Answer 6.8. $3x + 7y + 5z = 28$

Answer 6.9. $17/3$

Answer 6.10. $5x + 2y - 3z = 7$, $(111/38, 52/38, 131/38)$, $\sqrt{293/38}$

Answer 6.12. $B(123/50, -286/50, 255/50)$ and $59/\sqrt{50}$

Answer 6.14. $2x - 7y + 6z = 21$

Answer 6.16. $\mathbf{x} = (1, 1, 5) + t(1, 1, 5)$, $2\sqrt{3}$, 3