Dawson College: Linear Algebra: 201-NYC-05-S06: Fall 2015

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Test 1

This test is graded out of 48 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcl} 3x_1 & + & 3x_2 & + & 7x_3 & - & 3x_4 & + & x_5 & = & 3 \\ 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & - & 2x_5 & = & 1 \\ 4x_1 & & & + & 17x_3 & - & 2x_4 & - & x_5 & = & 1 \end{array}$$

b. *(1 mark)* Find two particular solution to the above system.

c. *(1 mark)* Find a solution to the above system where $x_3 = 1$.

a) $\left[\begin{array}{ccccc|c} 3 & 3 & 7 & -3 & 1 & 3 \\ 2 & 3 & 3 & 1 & -2 & 1 \\ 4 & 0 & 17 & -2 & -1 & 1 \end{array}\right]$

$\sim R_1 \leftrightarrow R_2 \left[\begin{array}{cccccc} 2 & 3 & 3 & 1 & -2 & 1 \\ 3 & 3 & 7 & -3 & 1 & 3 \\ 4 & 0 & 17 & -2 & -1 & 1 \end{array}\right]$

$\sim 2R_2 \to R_2 \left[\begin{array}{cccccc} 2 & 3 & 3 & 1 & -2 & 1 \\ 6 & 6 & 14 & -6 & 2 & 6 \\ 4 & 0 & 17 & -2 & -1 & 1 \end{array}\right]$

$\sim \begin{array}{l} -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{cccccc} 2 & 3 & 3 & 1 & -2 & 1 \\ 0 & -3 & 5 & -9 & 8 & 3 \\ 0 & -6 & 11 & -4 & 3 & -1 \end{array}\right]$

$\sim -2R_2 + R_3 \to R_3 \left[\begin{array}{cccccc} 2 & 3 & 3 & 1 & -2 & 1 \\ 0 & -3 & 5 & -9 & 8 & 3 \\ 0 & 0 & 1 & 14 & -13 & -7 \end{array}\right]$

$\sim \begin{array}{l} -3R_3 + R_1 \to R_1 \\ -5R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{cccccc} 2 & 3 & 0 & -41 & +37 & 22 \\ 0 & -3 & 0 & -79 & 73 & 38 \\ 0 & 0 & 1 & 14 & -13 & -7 \end{array}\right]$

$\to R_2 + R_1 \to R_1 \left[\begin{array}{cccccc} 2 & 0 & 0 & -120 & 110 & 60 \\ 0 & -3 & 0 & -79 & 73 & 38 \\ 0 & 0 & 1 & 14 & -13 & -7 \end{array}\right]$

$\sim \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -\frac{1}{3}R_2 \to R_2 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & -60 & 55 & 30 \\ 0 & 1 & 0 & 79/3 & -73/3 & -38/3 \\ 0 & 0 & 1 & 14 & -13 & -7 \end{array}\right]$

Let $x_4 = s$
$x_5 = t$

$\therefore x_1 = 60s - 55t + 30$
$x_2 = -\frac{79}{3}s + \frac{73}{3}t - \frac{38}{3}$
$x_3 = -14s + 13t - 7$
$x_4 = s$ $\quad s, t \in \mathbb{R}$
$x_5 = t$

b) let $s = t = 0$
$(x_1, x_2, x_3, x_4, x_5) = (30, -\frac{38}{3}, -7, 0, 0)$
let $t = 1$ $s = 0$
$(x_1, x_2, x_3, x_4, x_5) = (-25, \frac{35}{3}, 6, 0, 1)$

c) $1 = x_3$
$1 = -14s + 13t - 7$ let $t = 0$
$8 = -14s$
$-\frac{4}{7} = s$
$\therefore x_4 = \frac{4}{7}, x_5 = 0$
$x_1 = \frac{-30}{7}, x_2 = \frac{316}{21}$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -4 & 0 \end{bmatrix} C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & -1 \end{bmatrix} D = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} F = \begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

$\mathrm{tr}(BAF)$

$$BAF = B \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix} = B \begin{bmatrix} 11 & -2 \\ 0 & 2 \\ 17 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -3 & -4 & 0 \end{bmatrix}\begin{bmatrix} 11 & -2 \\ 0 & 2 \\ 17 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & -7 \\ -33 & -2 \end{bmatrix}$$

$$\mathrm{tr}(BAF) = 39 - 2 = 37$$

b. *(2 marks)* Compute the following, if possible.

CAB not defined

since C is 3×2 and A is 3×3

c. *(2 marks)* Compute the following, if possible.

$(FD^{-1})^T$

$$c)\ D^{-1} = \frac{1}{2(2) - 3(1)} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \right)^T = \begin{bmatrix} 11 & -17 \\ 2 & -4 \\ 2 & -3 \end{bmatrix}^T = \begin{bmatrix} 11 & 2 & 2 \\ -17 & -4 & -3 \end{bmatrix}$$

d. *(5 marks)* Find E, if possible.

$$(I - E^T)^{-1} = (\mathrm{tr}(D)D^2)^T$$

$$\left((I - E^T)^{-1}\right)^T = \left((\mathrm{tr}(D)D^2)^T\right)^T$$

$$\left((I - E^T)^T\right)^{-1} = \mathrm{tr}(D)D^2$$

$$\left(I^T - (E^T)^T\right)^{-1} = \mathrm{tr}(D)D^2$$

$$(I - E)^{-1} = 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$(I - E)^{-1} = 4 \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$I - E = \left(4 \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}\right)^{-1}$$

$$I - E = \frac{1}{4} \frac{1}{49 - 48} \begin{bmatrix} 7 & -12 \\ -4 & 7 \end{bmatrix}$$

$$E = -\begin{bmatrix} 7/4 & -3 \\ -1 & 7/4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} -3/4 & 3 \\ 1 & -3/4 \end{bmatrix}$$

Question 3. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 3 & 0 \\ 3 & 2 & 1/2 \end{bmatrix}.$$

a. *(5 marks)* Find A^{-1}.

b. *(3 marks)* Solve for X where $AX = B$ and

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

a) $[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -2R_1 + R_2 \to R_2 \\ 2R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\sim \; -3R_1 + R_3 \to R_3 \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 2 \end{array}\right]$$

$$\sim \; -2R_2 + R_3 \to R_3 \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{array}\right]$$

$$\sim \; 2R_2 + R_1 \to R_1 \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -3 & 2 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 2 \end{array}\right]$$

$$\sim \begin{array}{l} \frac{1}{2}R_1 \to R_1 \\ -R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -3/2 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

b) $AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

$$X = \begin{bmatrix} -3/2 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -4 & 2 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & 1 & -3/4 & 2 & -1 \\ 2 & -1 & 1 & -2 & 1 \\ -7 & 2 & 3/2 & -4 & 2 \end{bmatrix}$$

Question 4. *(4 marks)* Prove: Let A be a square matrix. If $A^4 = AAAA = 4I$ then A is invertible.

notice $\frac{1}{4}AAAA = I$

so $\left(\frac{1}{4}AAA\right)A = I$

and $A\left(\frac{1}{4}AAA\right) = I$

$\therefore A^{-1} = \frac{1}{4}AAA$

Question 5. *(4 marks)* Prove: $AB = BA$ if and only if $(A+B)(A-B) = (A-B)(A+B)$

[$\Rightarrow$] if $AB = BA$ then $LHS = A^2 - AB + BA - B^2 = A^2 - AB + AB - B^2$ by premise

$= A^2 - B^2$

$RHS = (A-B)(A+B) = A^2 - BA + AB - B^2$

$= A^2 - BA + BA - B^2$ by premise

$= A^2 - B^2$

[$\Leftarrow$] if $(A+B)(A-B) = (A-B)(A+B)$

$\cancel{A^2} - AB + BA - \cancel{B^2} = \cancel{A^2} + AB - BA - \cancel{B^2}$

$2BA = 2AB$

$BA = AB$

Question 6. *(5 marks)* Express

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \sim -R_1+R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

as a product of ~~5~~ 4 elementary matrices.

$$\sim -R_3+R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 \sim -R_1+R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$I_3 \sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_2$$

$$I_3 \sim -R_3+R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$I_4 \sim \tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4$$

$$E_4E_3E_2E_1A = I$$

So $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 7. The augmented matrix of a linear system is given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a^2-1 & b^2-a^2 \end{bmatrix}$$

If possible for what values of a and b there is

a. *(2 marks)* no solution? Justify.

b. *(2 marks)* exactly one solution? Justify.

c. *(2 marks)* infinitely many solutions? Justify.

a) no solution if $a = \pm 1$ and $a \neq \pm b$ then $0x_1 + 0x_2 + 0x_3 + \overbrace{(a^2-1)}^{0}x_4 \neq 0$
no (x_1, x_2, x_3, x_4) satisfy the last equation

b) Not possible since # leading 1's < # var.

c) if the system is consistent that is $a \neq \pm 1$ or $a = \pm b$.
(look at a, and b).

Bonus Question.*(5 marks)*
If A, B and $A+B$ are invertible matrices of the same size then show that $A^{-1}+B^{-1}$ is invertible and find a formula for $(A^{-1}+B^{-1})^{-1}$.

$(A+B)^{-1}(A+B) = I$ since $A+B$ is invertible

$(A+B)^{-1}(IA+BI) = I$

$(A+B)^{-1}(BB^{-1}A + BA^{-1}A) = I$ since A and B are invertible

$(A+B)B(B^{-1}+A^{-1})A = I$

$(A+B)B(B^{-1}+A^{-1})AA^{-1} = IA^{-1}$

$(A+B)B(B^{-1}+A^{-1})I = A^{-1}$

$A(A+B)B(B^{-1}+A^{-1}) = AA^{-1}$

$A(A+B)B(B^{-1}+A^{-1}) = I$

$A(A+B)B(A^{-1}+B^{-1}) = I$

So the left inverse of $A^{-1}+B^{-1}$ is

$A(A+B)B$

Similarly we can show the right inverse is $A(A+B)B$

$\therefore (A^{-1}+B^{-1})^{-1} = A(A+B)B$