Dawson College: Linear Algebra: 201-NYC-05-S06: Fall 2015

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Test 2

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & -4 \\ -1 & 2 & 2 \\ -3 & 4 & 6 \end{bmatrix}$$

a. *(4 marks)* Find the adj(A).

b. *(4 marks)* Evaluate $\det(B^{101}\operatorname{adj}(A) + BA^{2015})$.

c. *(2 marks)* Justify that the components of the solution of the system $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

are integers.

a)
$$A \sim \begin{matrix} -2R_1+R_2 \to R_2 \\ -3R_1+R_3\to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -3 \end{bmatrix} \sim_{-R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$\det A = \det B$? B

$= 1$

$C_{11} = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -2$ $\quad C_{12} = -\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$ $\quad C_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 3$ $\quad C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3$ $\quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$ $\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2$ $\quad C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$$\operatorname{adj}(A) = \begin{bmatrix} -2 & 0 & 1 \\ 3 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

b) $\det(B^{101}\operatorname{adj}(A) + BA^{2015})$

$= \det(B(B^{100}\operatorname{adj}(A) + A^{2015}))$

$= \det B \det(B^{100}\operatorname{adj}(A) + A^{2015})$

$= 0 \det(B^{100}\operatorname{adj}(A) + A^{2015})$

$= 0$

$\det B = 0$ since $C_3 = -2C_1$

c) $x_1 = \dfrac{|A_1|}{|A|}, \; x_2 = \dfrac{|A_2|}{|A|}, \; x_3 = \dfrac{|A_3|}{|A|}$

$= |A_1| \qquad = |A_2| \qquad = |A_3|$

$|A_i|$ are integer values since their entries are integers and determinants are product, sum and difference of the entries of the matrix.
And the product, sum and difference of integers is an integer.

Question 2. Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 0 & 2 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 1 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 2 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 2 & 3 & 3 & -3 \\ -3 & 3 & 1 & 9 \\ 0 & 1 & -1 & -12 \end{bmatrix} \underset{3R_1+R_3\to R_3}{\overset{-2R_1+R_2\to R_2}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & -1 & -15 \\ 0 & 6 & 7 & 27 \\ 0 & 1 & -1 & -12 \end{bmatrix}$$

$$\underset{-R_2+R_4\to R_4}{\overset{-6R_2+R_3\to R_3}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & -1 & -15 \\ 0 & 0 & 13 & 117 \\ 0 & 0 & 0 & 3 \end{bmatrix} = C$$

a. *(4 marks)* Evaluate $\det(B)$.

b. *(4 marks)* If C is a square matrix then determine $\det(\frac{\det(B)}{2}C^TA^3) = \pi$, if possible.

$$\det B = \det C = 13(3) = 39$$

b) $\det\left(\frac{\det B}{2} C^T A^3\right) = \pi$

$$\left(\frac{\det B}{2}\right)^{10} \det C^T \det A^3 = \pi$$

$$\left(\frac{39}{2}\right)^{10} \det C (\det A)^3 = \pi$$

$$\left(\frac{39}{2}\right)^{10} \det C \,(2^5)^3 = \pi$$

$$\det C = \frac{\pi}{2^{15}} \frac{2^{10}}{39^{10}} = \frac{\pi}{2^5\, 39^{10}}$$

Question 3. *(2 marks)* A non-zero square matrix A is said to be *nilpotent of degree 2* if $A^2 = 0$. Prove or disprove: There exists a square 2×2 matrix that is symmetric and nilpotent of degree 2.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+cb & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A^T = A$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$b = c$

$a^2 + cb = 0$ $\quad ab + bd = 0$

$ac + cd = 0$ $\quad bc + d^2 = 0$

sub $b = c$

① $a^2 + c^2 = 0$ $\quad ac + bc = 0$

$ac + cd = 0$ $\quad$ ② $c^2 + d^2 = 0$

① $\Rightarrow a = 0$ and $c = 0$

② $\Rightarrow d = 0 \Rightarrow b = 0$

$\therefore A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Question 4. [1] Let

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$$

and

$$B = \begin{bmatrix} a+2b+4c & d+2e+4f & g+2h+4k \\ 3a+4b+7c & 3d+4e+7f & 3g+4h+7k \\ 5a+7b+8c & 5d+7e+8f & 5g+7h+8k \end{bmatrix}$$

a. *(3 marks)* Find a matrix C such that $B = CA$.

b. *(2 marks)* Find the value of λ such that $\det(B) = \lambda \det(A)$ for all possible choices of A.

a)

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & K \end{bmatrix}$$

$$\det(B) = \det(CA)$$
$$= \det C \det A$$

b)

$$\det C = 1 \begin{vmatrix} 4 & 7 \\ 7 & 8 \end{vmatrix} - 2 \begin{vmatrix} 3 & 7 \\ 5 & 8 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}$$
$$= 32-49 - 2[24-35] + 4[21-20]$$
$$= -17 - 2[-11] + 4$$
$$= 9$$

$$\therefore \lambda = 9$$

Question 5. [2] *(5 marks)* Suppose $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ such that $||\vec{u}|| = 3$, $||\vec{v}|| = 5$, and $||\vec{u}+\vec{v}|| = 7$. Find the angle between $\vec{u}$ and $\vec{v}$.

$$||\vec{u} + \vec{v}|| = 7$$
$$7^2 = ||\vec{u}+\vec{v}||^2$$
$$49 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$49 = \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$49 = ||\vec{u}||^2 + 2\vec{u}\cdot\vec{v} + ||\vec{v}||^2$$
$$49 = 3^2 + 2\vec{u}\cdot\vec{v} + 5^2$$
$$\frac{15}{2} = \vec{u}\cdot\vec{v}$$

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta$$
$$\frac{15}{2} = 3(5)\cos\theta$$
$$\frac{1}{2} = \cos\theta$$
$$\theta = \pi/3$$

[1] From a John Abbott final examination

[2] From a John Abbott final examination

Question 6. *(5 marks)* Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{0}$ has just the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has just the trivial solution.

[$\Rightarrow$] Premise: $Ax=0$ has only the trivial solution.

Conclusion: $(QA)x=0$ has only the trivial solution

By the TFAE the premise implies that A is invertible.
Since Q is invertible then QA is invertible.
$\therefore$ by TFAE $(QA)x=0$ has only the trivial solution.

[$\Leftarrow$] Premise: $(QA)x=0$ has only the trivial solution

Conclusion: $Ax=0$ has only the trivial solution.

By the TFAE the premise implies that QA is invertible.
So A is invertible $\therefore$ by TFAE $Ax=0$ has only the trivial solution.

Question 7. Given $\vec{u} = (1,-2,3)$, $\vec{v} = (3,2,4)$, and $\vec{w} = (-4,1,-5)$.

a. *(2 marks)* Find a unit vector that is oppositely directed to $\vec{u}$.

b. *(2 marks)* Compute $\|-\|\vec{w}\|\vec{u}\|$, if possible.

c. *(2 marks)* Compute $(\vec{u}\cdot\vec{v}) - \vec{w}$, if possible.

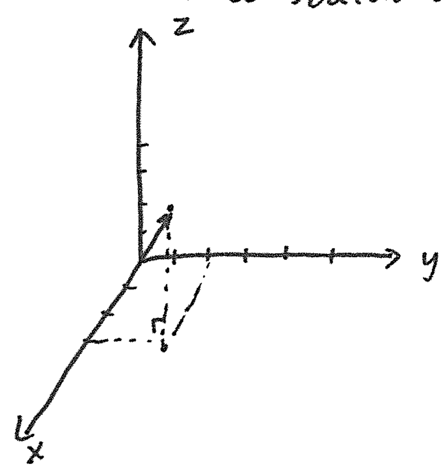
d. *(2 marks)* Sketch $\vec{v}$.

a) $-\dfrac{\vec{u}}{\|\vec{u}\|} = -\dfrac{(1,-2,3)}{\sqrt{1^2+(-2)^2+3^2}} = \left(\dfrac{-1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$

b) $\|\vec{w}\| = \sqrt{(-4)^2+1^2+(-5)^2} = \sqrt{42}$

$\|-\|\vec{w}\|\vec{u}\| = \|-\sqrt{42}\,\vec{u}\| = \sqrt{42}\,\|\vec{u}\| = \sqrt{42}\sqrt{14} = \sqrt{14}\sqrt{42}$

c) impossible since $\vec{u}\cdot\vec{v}$ is a scalar and $\vec{w}$ is a vector.



Bonus Question.*(5 marks)*
Let A and B denote invertible $n \times n$ matrices. Show that:

$$\operatorname{adj}(AB) = (\operatorname{adj}(A))(\operatorname{adj}(B))$$

A & B invertible $\Rightarrow$ AB is invertible

$$\therefore \quad (AB)^{-1} = \frac{1}{\det AB} \operatorname{adj}(AB)$$

and $$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$B^{-1} = \frac{1}{\det B} \operatorname{adj} B$$

In addition $$(AB)^{-1} = B^{-1}A^{-1}$$

$$\frac{1}{\det(AB)} \operatorname{adj}(AB) = \frac{1}{\det B} \operatorname{adj} B \frac{1}{\det A} \operatorname{adj} A$$

$$\operatorname{adj}(AB) = \det(AB) \frac{1}{\det B} \operatorname{adj} B \frac{1}{\det A} \operatorname{adj} A$$

$$\operatorname{adj}(AB) = \cancel{\det A} \cancel{\det B} \frac{1}{\cancel{\det B}} \frac{1}{\cancel{\det A}} \operatorname{adj} B \operatorname{adj} A$$

$$\operatorname{adj}(AB) = \operatorname{adj} B \operatorname{adj} A$$