Dawson College: **Linear Algebra**: 201-NYC-05-S06: Fall 2015

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

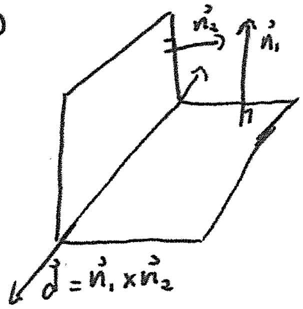
$$\begin{array}{lrcll} \mathscr{A}: & (-3,-2,-1) & & & \\ \mathscr{L}_1: & (x,y,z) & = & (2+t,1-t,3t) & t\in\mathbb{R} \\ \mathscr{L}_2: & (x,y,z) & = & (2t,-3+t,2+2t) & t\in\mathbb{R} \\ \mathscr{P}_1: & x+2y+3z & = & 10 & \\ \mathscr{P}_2: & -3x-2y+z & = & 21 & \end{array}$$

a. *(3 marks)* Find an equation for the line parallel to both $\mathscr{P}_1$ and $\mathscr{P}_2$ containing $\mathscr{A}$.

b. *(4 marks)* Are $\mathscr{L}_1$ and $\mathscr{L}_2$ skew lines?

c. *(3 marks)* Are $\mathscr{L}_2$ and $\mathscr{P}_2$ parallel, perpendicular, or neither? Do they intersect? If so find the point(s) of intersection.

a)



$$\vec{d} = \vec{n}_1 \times \vec{n}_2 = \left(\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix} \right) = (8, -10, 4)$$

$$\begin{array}{cc} \vec{n}_1 & \vec{n}_2 \\ 1 & -3 \\ 2 & -2 \\ 3 & 1 \end{array}$$

$$(x,y,z) = A + t\vec{d}$$
$$= (-3,-2,-1) + t(8,-10,4)$$

b) $\mathscr{L}_1 \nparallel \mathscr{L}_2$ since $\vec{d}_1 \neq k\vec{d}_2$ where $\vec{d}_1 = (1,-1,3)$ and $\vec{d}_2 = (2,1,2)$.

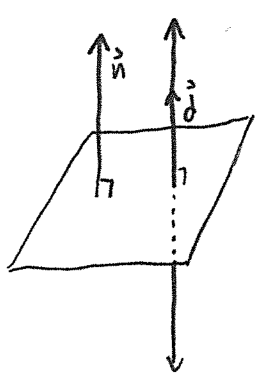
Does $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect?

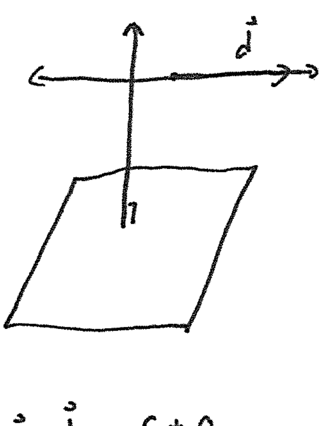
① $2+t = 2s$
② $1-t = -3+s$
③ $3t = 2+2s$

①+② $3 = -3+3s$
$s = 2$
sub into ①
$2+t = 2(2)$
$t = 2$
sub $s=2$, $t=2$ into ③
$3(2) \stackrel{?}{=} 2+2(2)$
$6 = 6$

∴ $\mathscr{L}_1$ and $\mathscr{L}_2$ are not skew since they intersect

c)





$\vec{n} = (-3,-2,1)$
$\vec{d} = (2,1,2)$
$\vec{n} \neq k\cdot\vec{d}$ ∴ $\mathscr{L}_2 \not\perp P_2$

$\vec{n}\cdot\vec{d} = -6 \neq 0$
∴ $\mathscr{L}_2 \nparallel P_2$

∴ $\mathscr{L}_2$ and P_2 intersect,
sub $\mathscr{L}_2$ into P_2

$$-3(2t) - 2(-3+t) + 2+2t = 21$$
$$-6t = 13$$
$$t = \frac{-13}{6}$$

∴ point of intersection

$$(x,y,z) = \left(2\left(\tfrac{-13}{6}\right), -3 - \tfrac{13}{6}, 2+2\left(\tfrac{-13}{6}\right)\right)$$
$$= \left(\frac{-13}{3}, \frac{-31}{6}, \frac{-7}{3}\right).$$

Question 2. *(4 marks)* Show that if $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = \vec{0}$. Give a geometrical interpretation of the result.

$$\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}\vec{v})$$

$$= \text{proj}_{\vec{u}}\left(\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}\right)$$

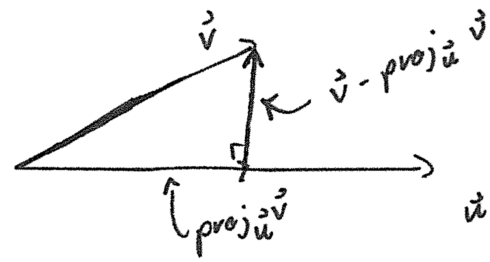
$$= \frac{\left(\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}\right)\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\cdot\vec{u}$$

$$= \frac{\left(\vec{v}\cdot\vec{u} - \frac{\vec{u}\cdot\vec{v}\,\cancel{\vec{u}\cdot\vec{u}}}{\cancel{\vec{u}\cdot\vec{u}}}\right)}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{\vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{0}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \vec{0}$$



The vector $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ is $\perp$ $\vec{u}$.
$\therefore$ the orthogonal projection of $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$ onto $\vec{u}$ is $\vec{0}$

Question 3. *(4 marks)* Find all value(s) of y for which the parallelepiped generated by the vectors $(1,y,3)$, $(2,1,3)$ and $(-1,-2,1)$ has a volume of 13.

$$13 = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

$$13 = \left|\begin{vmatrix} 1 & y & 3 \\ 2 & 1 & 3 \\ -1 & -2 & 1 \end{vmatrix}\right|$$

$$13 = \left|\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}\right|$$

$$13 = |1 + 6 - y(2+3) + 3(-4+1)|$$

$$13 = |7 - 5y + 3(-3)|$$

$$13 = |-2 - 5y|$$

$13 = -2 - 5y$ and $-13 = -2 - 5y$

$y = -3$ $\qquad$ $\frac{11}{5} = y$

Question 4. Given the following system

$x+z=1$

$y+z=1$

associated homogeneous system $\begin{matrix} x & & +z=0 \\ & y & +z=0 \end{matrix}$

a. *(2 marks)* Express a general solution of this system as a particular solution of the system plus a general solution of the associated homogeneous system.

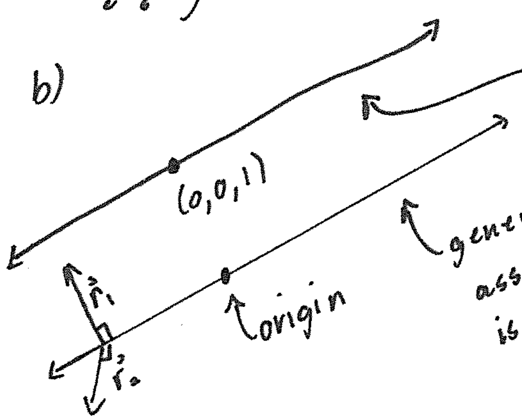
b. *(2 marks)* Give a geometric interpretation of the result of part a. Discuss the general solution geometrically with respect to the associated homogeneous system.

a) $\left[\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}\right]$ Let $z=t$ then $\begin{matrix} x=-t \\ y=-t \end{matrix}$ $\therefore (x,y,z)=(-t,-t,t)=t(-1,-1,1)$ $t\in\mathbb{R}$

and a particular solution to the system is $(0,0,1)$

$\therefore$ general solution is $(x,y,z)=(0,0,1)+t(-1,-1,1)$ $t\in\mathbb{R}$.

b)



general solution of the associated homogeneous system is $\perp$ $\vec{r}_1=(1,0,1)$ and $\vec{r}_2=(0,1,1)$

general solution of the system is that the sol. of the associated hom. system translated by $(0,0,1)$.

Question 5.[1] *(5 marks)* Given that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three linearly independent vectors in $\mathbb{R}^n$. For which value(s) of k will the vectors $\vec{u}+2\vec{v}$, $\vec{v}+3\vec{w}$ and $k\vec{u}+\vec{w}$ be linearly dependent?

$\vec{0}=c_1(\vec{u}+2\vec{v})+c_2(\vec{v}+3\vec{w})+c_3(K\vec{u}+\vec{w})$

$\vec{0}=(c_1+Kc_3)\vec{u}+(2c_1+c_2)\vec{v}+(3c_2+c_3)\vec{w}$

Since $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly independent

$c_1+Kc_3=0$

$2c_1+c_2=0$

$3c_2+c_3=0$

$$\underbrace{\begin{bmatrix} 1 & 0 & K \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only trivial solution iff $|A|\neq 0$

$|A|=\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}+K\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}$

$=1+6K$

$|A|\neq 0$ iff $K\neq -\frac{1}{6}$

$\therefore$ the set is linearly independent if $K\neq -\frac{1}{6}$

[1] From a John Abbott final examination

Question 6. Let $V = \{A \mid A \in \mathscr{M}_{2\times 2} \text{ and } \det(A) \neq 0\}$ with the following operations:

$A + B = AB$ and $kA = kA$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication

a. *(2 marks)* Does V satisfy closure under vector addition? Justify.

b. *(2 marks)* Does V contain a zero vector? If so find it. Justify.

c. *(2 marks)* Does V contain an additive inverse for all of its vectors? Justify.

d. *(2 marks)* Does V satisfy closure under scalar multiplication? Justify.

a) Let $A, B \in V$ then $\det(A) \neq 0$, $\det(B) \neq 0$

$A + B \in V$ since $\det(A+B)$

$= \det(AB)$

$= \det A \det B \neq 0$ $\therefore$ closed under vector addition

b) Let $\vec{0} \in M_{2\times 2}$ then for any $A \in V$

$A + \vec{0} = A$

$A\vec{0} = A$

$A^{-1}A\vec{0} = A^{-1}A$ since $\det(A) \neq 0$ and $A \in V$

$\vec{0} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \vec{0} = I \in V$

c) Let $A \in V$ and $B \in M_{2\times 2}$ then

$A + B = \vec{0}$

$AB = I$

$A^{-1}AB = A^{-1}I$ since $\det(A) \neq 0$ and $A \in V$

$B = A^{-1} \in V$ since $\det(A^{-1}) \neq 0$

$\therefore$ V contains an additive inverse for all of its vectors.

d) Let $A \in V$ then $0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\det(0A) = 0$

$\therefore 0A \notin V$

$\therefore$ not closed under scalar multiplication.

Question 7. *(5 marks)* Given the following two subspace of $\mathbb{R}^3$: $W_1 = \{x \mid x \in \mathbb{R}^3 \text{ and } A_1x = 0\}$ and $W_2 = \{x \mid x \in \mathbb{R}^3 \text{ and } A_2x = 0\}$ where

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -3 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} 5 & 7 & 9 \\ -5 & -7 & -9 \\ 10 & 14 & 18 \end{bmatrix}$$

Determine whether the two subspace are equal or a subspace is contained in the other. Hint: For each subspace determine a set of vectors that spans it.

$A_1 x = 0$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ -3 & -3 & -3 & 0 \end{array}\right] \sim \begin{array}{c} -4R_1+R_2 \to R_2 \\ 3R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \sim \begin{array}{c} -\frac{1}{3}R_2 \to R_2 \\ R_2+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim -2R_2+R_1 \to R_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $z = t$ then $x = t$
$y = -2t$
$\therefore (x,y,z) = t(1,-2,1)$

$\therefore W_1 = \text{span}(\{(1,-2,1)\})$

$A_2 x = 0$

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ -5 & -7 & -9 & 0 \\ 10 & 14 & 18 & 0 \end{bmatrix} \sim \begin{array}{c} R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $y = s$, $z = t$
$x = -\frac{7}{5}s - \frac{9}{5}t$
$\therefore (x,y,z) = s(-\frac{7}{5}, 1, 0) + t(-\frac{9}{5}, 0, 1)$
$= s'(-7,5,0) + t'(-9,0,5)$

$\therefore W_2 = \text{span}(\{(-7,5,0),(-9,0,5)\})$

$(-7,5,0) \notin \text{span}(\{(1,-2,1)\})$
and $(-9,0,5) \notin \text{span}(\{(1,-2,1)\})$ $\therefore W_2 \nsubseteq W_1$

Is $(1,-2,1) \in \text{span}(\{(-7,5,0),(-9,0,5)\})$?

$(1,-2,1) = c_1(-7,5,0) + c_2(-9,0,5)$

① $1 = -7c_1 - 9c_2$
② $-2 = 5c_1$ $\quad c_1 = -2/5$
③ $1 = 5c_2$ $\quad c_2 = 1/5$

sub into ① $\quad 1 \overset{?}{=} -7(\frac{-2}{5}) - 9(\frac{1}{5})$
$1 \overset{?}{=} \frac{5}{5} = 1$

$\therefore (1,-2,1) \in \text{span}(\{(-7,5,0),(-9,0,5)\})$

$\therefore W_1 \subseteq W_2$

Bonus Question. *(5 marks)*
For n real-valued functions $f_1, \ldots, f_n$, which are $n-1$ times differentiable on an interval I, the *Wronskian* $W(f_1, \ldots, f_n)$ as a function on I is defined by

$$W(f_1,\ldots,f_n)(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}, \quad x \in I.$$

Theorem: If the n real-valued functions $f_1, \ldots, f_n$, which are $n-1$ times differentiable on an interval I, and the Wronskian of these functions is not identically zero on I, then these functions form a linearly independent set of vectors on I.

Determine whether the set $\{1, e^x, \arctan x\}$ is linearly independent on $\mathbb{R}^+$

$W(1, e^x, \arctan x)$

$$= \begin{vmatrix} 1 & e^x & \arctan x \\ 0 & e^x & \dfrac{1}{x^2+1} \\ 0 & e^x & \dfrac{-2x}{(x^2+1)^2} \end{vmatrix}$$

$$= \frac{-2e^x x}{(x^2+1)^2} + \frac{e^x}{x^2+1}$$

$$= \frac{-e^x (x+1)^2}{(x^2+1)^2} \neq 0 \text{ on } \mathbb{R}^+$$

$\therefore \{1, e^x, \arctan x\}$ is linearly independent on $\mathbb{R}^+$