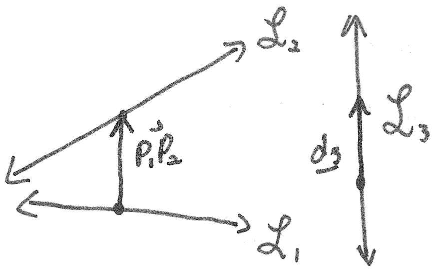
Dawson College: Fall 2019: Linear Algebra (Science): 201-NYC-05-S5: **Quiz 11** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Given the following lines which are all skew to each other:

$$\begin{array}{lcl} \mathcal{L}_1 & : & (x,y,z) = (1,0,0) \; + \; t_1(1,2,0) \\ \mathcal{L}_2 & : & (x,y,z) = (1,1,0) \; + \; t_2(1,0,1) \\ \mathcal{L}_3 & : & (x,y,z) = (1,0,1) \; + \; t_3(1,2,3) \end{array}$$

here $t_1,\ t_2,\ t_3 \in \mathbb{R}$. Consider a line $\mathcal{L}_4$ that is parallel to $\mathcal{L}_3$ and intersects both $\mathcal{L}_1$ and $\mathcal{L}_2$. Find the points of intersection of $\mathcal{L}_4$ with $\mathcal{L}_1$ and $\mathcal{L}_4$ with $\mathcal{L}_2$.



So. $\vec{P_1P_2} = k\,\vec{d_3}$

where $\vec{P_1P_2} = (1+t_2, 1, t_2) - (1+t_1, 2t_1, 0)$

$= (t_2 - t_1, 1-2t_1, t_2)$

$(t_2 - t_1, 1-2t_1, t_2) = k(1,2,3)$

we obtain

$$\begin{cases} t_2 - t_1 = k \\ 1-2t_1 = 2k \\ t_2 = 3k \end{cases}$$

$$\begin{cases} t_1 - t_2 + k = 0 \\ 2t_1 + 2k = 1 \\ -t_2 + 3k = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & -1 & 3 & 0 \end{bmatrix}$$

$$\sim \; -2R_1 + R_2 \to R_2 \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & 3 & 0 \end{bmatrix}$$

$$\begin{array}{l} \tfrac{1}{2}R_2 + R_1 \to R_1 \\ \sim \\ \tfrac{1}{2}R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 1 & \tfrac{1}{2} \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & \tfrac{1}{2} \end{bmatrix}$$

$$\begin{array}{l} -\tfrac{1}{3}R_3 + R_1 \to R_1 \\ \sim \; \tfrac{1}{2}R_2 \to R_2 \\ \tfrac{1}{3}R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & \tfrac{1}{3} \\ 0 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & 1 & \tfrac{1}{6} \end{bmatrix}$$

$\therefore\ t_1 = \tfrac{1}{3}$

$t_2 = \tfrac{1}{2}$

$\therefore$ points of intersection are

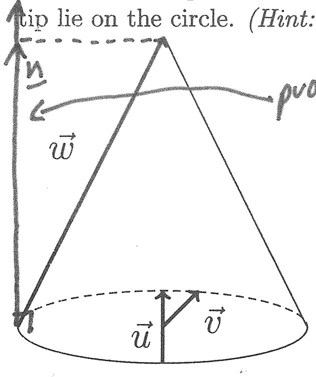
$\mathcal{L}_1 : (x,y,z) = (1,0,0) + \tfrac{1}{3}(1,2,0)$

$= \left(\tfrac{4}{3}, \tfrac{2}{3}, 0\right)$

$\mathcal{L}_2 : (x,y,z) = (1,1,0) + \tfrac{1}{2}(1,0,1)$

$= \left(\tfrac{3}{2}, 1, \tfrac{1}{2}\right)$.

Question 2. *(5 marks)* Given the cone defined by the given vectors $\vec{u} = (2, 2, 4)$, $\vec{v} = (1, 2, 1)$ and $\vec{w} = (4, 1, 3)$. Find the volume of the cone. Note from the diagram that $\vec{w}$ is not perpendicular to the base, that $\vec{v}$ is positioned such that its tail is at the center of the circle and its tip lies on the circle, that $\vec{u}$ is positioned such that the vector passes through the center of the circle while its tail and tip lie on the circle. *(Hint: the volume of a cone is equal to one third of the area of the base times the height.)*



$$\text{proj}_{\underline{n}}\,\underline{w} = \frac{\underline{n}\cdot\underline{w}}{\underline{n}\cdot\underline{n}}\,\underline{n}$$

$$= \frac{(6,-2,-2)\cdot(4,1,3)}{(6,-2,-2)\cdot(6,-2,-2)}(6,-2,-2)$$

$$= \frac{24-2-6}{36+4+4}(6,-2,-2)$$

$$= \frac{16}{44}(6,-2,-2) = \frac{16\cdot 2}{44}(3,-1,-1)$$

$$= \frac{8}{11}(3,-1,-1)$$

where $\underline{n} = \underline{v}\times\underline{u} = \left(\begin{vmatrix}2&2\\1&4\end{vmatrix}, -\begin{vmatrix}1&2\\1&4\end{vmatrix}, \begin{vmatrix}1&2\\2&2\end{vmatrix}\right)$

$\begin{matrix}1&2\\2&2\\1&4\end{matrix} = (6,-2,-2)$

$\|\underline{v}\| = \|(1,2,1)\| = \sqrt{6}$

$$V = \frac{1}{3}\,\text{base} \times \text{height}$$

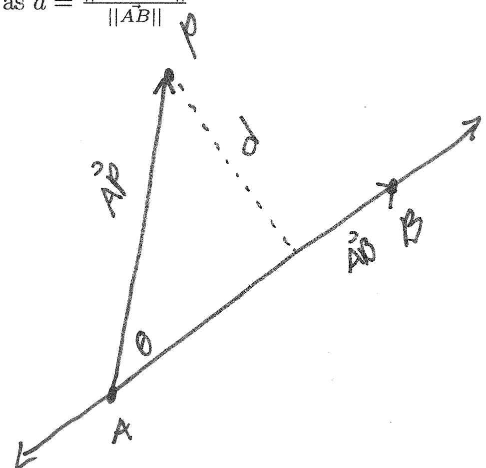
$$= \frac{1}{3}\pi r^2 \times \text{height}$$

$$= \frac{1}{3}\pi \|\underline{v}\|^2 \|\text{proj}_{\underline{n}}\,\underline{w}\|$$

$$= \frac{1}{3}\pi(\sqrt{6})^2 \left\|\frac{8}{11}(3,-1,-1)\right\|$$

$$= \pi\,\frac{1}{3}\cdot 6\cdot\frac{8}{11}\sqrt{11} = \frac{16\pi\sqrt{11}}{11}$$

Question 3. *(4 marks)* Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as $d = \frac{\|\vec{AP}\times\vec{AB}\|}{\|\vec{AB}\|}$



$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$$

$$\sin\theta = \frac{d}{\|\vec{AP}\|}$$

$$d = \|\vec{AP}\|\sin\theta$$

$$= \frac{\|\vec{AP}\times\vec{AB}\|}{\|\vec{AB}\|} \text{ from } ✱$$

$\|\vec{AP}\times\vec{AB}\| = \|\vec{AP}\|\|\vec{AB}\|\sin\theta$ ✱

Question 4. *(2 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, where $\vec{u}$ is nonzero and $\vec{u}\times\vec{v} = \vec{u}\times\vec{w}$, then $\vec{v} = \vec{w}$.

False,

Let $\underline{u} = (1,0,0)$

$\underline{v} = (2,0,0)$

$\underline{w} = (3,0,0)$

Then $\underline{u}\times\underline{v} = \underline{u}\times(2\underline{u}) = 2(\underline{u}\times\underline{u}) = 2\underline{0} = \underline{0}$

$\underline{u}\times\underline{w} = \underline{u}\times(3\underline{u}) = 3(\underline{u}\times\underline{u}) = 3\underline{0} = \underline{0}$

So $\underline{u}\times\underline{v} = \underline{u}\times\underline{w}$ but $\underline{v}\neq\underline{w}$