Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S7: Fall 2022: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Given A, an $n \times n$ matrix such that $\det(A) = 9$ and

$$A^3 A^T = 3A^{-1}\operatorname{adj}(A)$$

find n.

$$\det(A^3A^T) = \det(3A^{-1}\operatorname{adj}(A))$$
$$\det(A^3)\det(A^T) = 3^n \det(A^{-1})\det(\operatorname{adj}(A))$$
$$(\det A)^3 \det(A) = 3^n \frac{1}{\det(A)} (\det(A))^{n-1}$$
$$1 = 3^n \frac{1}{(\det(A))^5} (\det(A))^{n-1}$$
$$1 = 3^n (\det A)^{n-6}$$
$$1 = 3^n (3^2)^{n-6}$$
$$1 = 3^n 3^{2n-12}$$
$$1 = 3^{3n-12}$$
$$3^0 = 3^{3n-12}$$
$$0 = 3n-12$$
$$12 = 3n$$
$$n = 4$$

Question 2. *(3 marks)* Using Cramer's Rule find x_1 and x_3 for $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} \sin\theta & -\cos\theta & \sin\theta \\ \cos\theta & \sin\theta & \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2\sin\theta \\ 2\cos\theta \\ 2 \end{bmatrix}$.

$$|A| = \overbrace{a_{31}C_{31} + a_{32}C_{32}}^{0} + a_{33}C_{33} = 1(-1)^{3+3}\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta + \cos^2\theta = 1$$

$$|A_1| = \begin{vmatrix} 2\sin\theta & -\cos\theta & \sin\theta \\ 2\cos\theta & \sin\theta & \cos\theta \\ 2 & 0 & 1 \end{vmatrix} = 0 \text{ since } C_1 = 2C_3$$

$$\therefore x_1 = \frac{|A_1|}{|A|} = \frac{0}{1} = 0$$

$$|A_3| = \begin{vmatrix} \sin\theta & -\cos\theta & 2\sin\theta \\ \cos\theta & \sin\theta & 2\cos\theta \\ 0 & 0 & 2 \end{vmatrix} \underset{\frac{1}{2}C_3 \to C_3}{=} 2|A| = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{2}{1} = 2$$

Question 3. Determine whether the following statements are true or false for any $n \times n$ matrices A and B. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

1. *(3 marks)* There is no 3×3 matrix for which $A^2 + I_3 = 0$.

True, Suppose there exist a matrix s.t. $A^2 + I_3 = 0$.

Then

$$A^2 = -I_3$$
$$\det(A^2) = \det(-I_3)$$
$$(\det A)^2 = (-1)^3 \det(I_3)$$
$$(\det A)^2 = -1 \quad ↯$$

$\therefore$ such matrix does not exist.

Bonus Questions. *(5 marks)* Show that the following two statements are equivalent:

S1. P, Q, and $P+Q$ are all invertible and $(P+Q)^{-1} = P^{-1} + Q^{-1}$

S2. P is invertible and $Q = PG$ where $G^2 + G + I = 0$.