Dawson College: Calculus II: 201-NYB-05-S1: Summer 2008

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Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the average of the function $f(x) = x\sin x^2$ over the interval $[-\sqrt{\pi}, \sqrt{\pi}]$

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\sqrt{\pi}--\sqrt{\pi}}\int_{-\sqrt{\pi}}^{\sqrt{\pi}} x\sin x^2\,dx$$

$$= \frac{1}{2\sqrt{\pi}}\,0$$

$$= 0$$

Lets show that $f(x)$ is odd

$$f(-x) = -x\sin(-x)^2$$
$$= -x\sin x^2$$
$$= -f(x)$$

Question 2. *(5 marks)* Evaluate the following definite integral:

$$\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2 \frac{1}{(x-1)^2+1}\,dx$$

Complete the square

$$x^2-2x+2$$
$$= x^2-2x+1-1+2$$
$$= (x-1)^2+1$$

① $u = x-1$
② $du = dx$
$u(0) = 0-1 = -1$
$u(2) = 2-1 = 1$

$$\overset{①②}{=} \int_{-1}^{1} \frac{1}{u^2+1}\,du$$

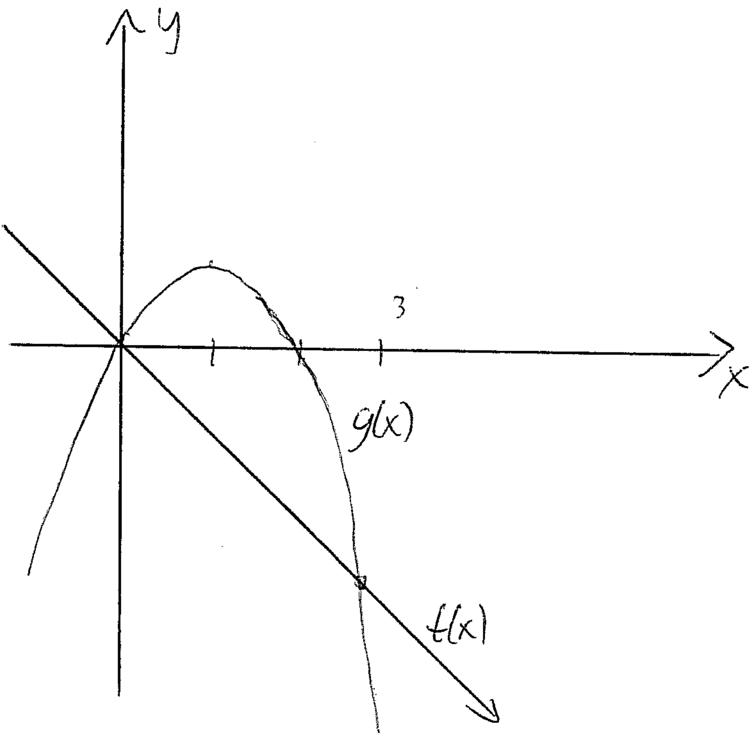
$$= \left[\arctan u\right]_{-1}^{1}$$

$$= \arctan 1 - \arctan -1$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

Question 3. *(2 marks for sketch and 3 marks for area)* Sketch the graph of the two following algebraic functions $f(x) = -x$ and $g(x) = -x^2 + 2x$ and find the area bounded by the two functions.



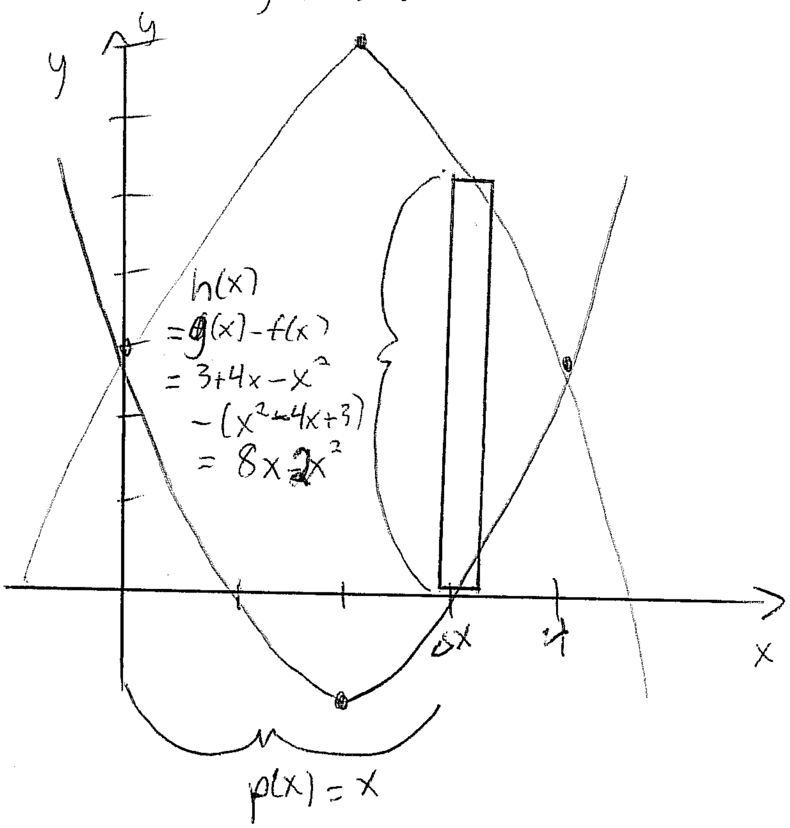
Lets find the intersection of the two curves

$$f(x) = g(x)$$
$$-x = -x^2 + 2x$$
$$0 = -x^2 + 3x$$
$$0 = -x(x-3)$$

$\therefore$ intersection at $x=0$ and $x=3$

$$V = \int_0^3 g(x) - f(x)\,dx$$
$$= \int_0^3 -x^2 + 2x - (-x)\,dx$$
$$= \int_0^3 -x^2 + 3x\,dx$$
$$= \left[\frac{-x^3}{3} + \frac{3x^2}{2}\right]_0^3$$
$$= \left[\frac{-3^3}{3} + \frac{3(3^2)}{2}\right]$$
$$= -9 + \frac{27}{2} = \frac{9}{2}$$

Question 4. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the function $f(x) = x^2 - 4x + 3$
$g(x) = 3 + 4x - x^2$ revolved about the y-axis.



Lets determine where both curve intersect.

$$f(x) = g(x)$$

$$x^2 - 4x + 3 = 3 + 4x - x^2$$

$$2x^2 - 8x = 0$$

$$2x(x-4) = 0$$

$\therefore$ intersection at $x=0$ and $x=4$

representative element:

$$\Delta V = 2\pi p(x) h(x) \Delta x$$

$$= 2\pi x (8x - x^2) \Delta x$$

$$= 2\pi (8x^2 - x^3) \Delta x$$

$$V = \int_0^4 2\pi (8x^2 - 2x^3) dx$$

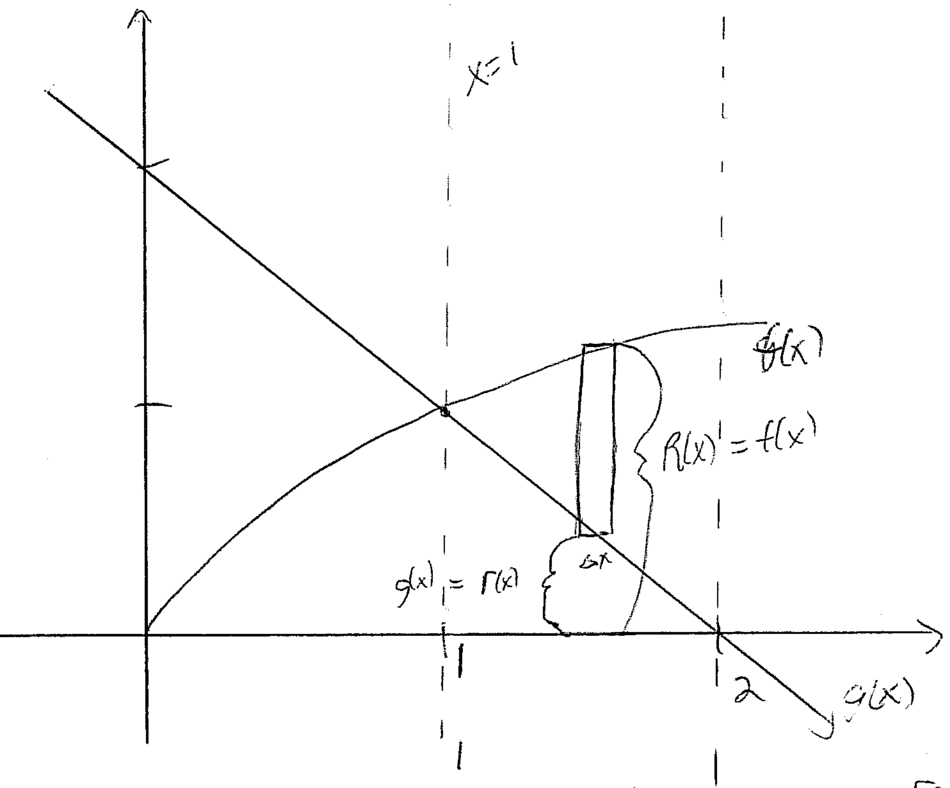
$$= 2\pi \left[\frac{8x^3}{3} - \frac{2x^4}{4} \right]_0^4$$

$$= 2\pi \left[\frac{8(4)^3}{3} - \frac{2(4^4)}{4} \right]$$

$$= 2\pi \left[\frac{128}{3} \right]$$

$$= \frac{256\pi}{3}$$

Question 5. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x) = \sqrt{x}$, $g(x) = 2 - x$ and the relations $x=1$ an' $x=2$ about the x-axis.



Lets find the intersection between the two curves

$f(x) = g(x)$

$\sqrt{x} = -x + 2$ sbs

$x = (-x+2)^2$

$x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4$

$0 = (x-1)(x-4)$

∴ only valid intersection at $x=1$ and $x=4$

<u>representative element:</u>

$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$
$$= \pi\left[(\sqrt{x})^2 - (2-x)^2\right]\Delta x$$
$$= \pi\left[x - 4 + 4x - x^2\right]\Delta x$$
$$= \pi\left[5x - 4 - x^2\right]\Delta x$$

$$V = \int_1^2 \pi\left[5x - 4 - x^2\right]dx$$

$$= \pi\left[\frac{5x^2}{2} - 4x - \frac{x^3}{3}\right]_1^2$$

$$= \pi\left[\left[\frac{5(2^2)}{2} - 4(2) - \frac{2^3}{3}\right] - \left[\frac{5(1)^2}{2} - 4(1) - \frac{1}{3}\right]\right]$$

$$= \frac{7\pi}{6}$$

Question 6. *(5 marks)* Find the arc length of the graph of the function $f(x) = \frac{3}{2}x^{2/3}+5$ over the interval $[0,8]$.

$$S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$f'(x) = x^{-1/3}$$

$$= \int_0^8 \sqrt{1+\left(\frac{1}{x^{1/3}}\right)^2}\,dx$$

$$= \int_0^8 \sqrt{1+\frac{1}{x^{2/3}}}\,dx$$

$$= \int_0^8 \sqrt{\frac{x^{2/3}+1}{x^{2/3}}}\,dx$$

$$u = x^{2/3}+1$$

$$du = \frac{2\,dx}{3x^{1/3}}$$

$$= \int_0^8 \sqrt{x^{2/3}+1}\,\frac{dx}{x^{1/3}}$$

$$\frac{3du}{2} = \frac{dx}{x^{1/3}}$$

$$u(0) = 1$$

$$u(8) = 5$$

$$= \int_1^5 \sqrt{u}\,\frac{3du}{2}$$

$$= \frac{3}{2}\int_1^5 \sqrt{u}\,du$$

$$= \frac{\cancel{3}}{\cancel{2}}\left[\frac{\cancel{2}u^{3/2}}{\cancel{3}}\right]_1^5$$

$$= \sqrt{125} - 1$$

Question 7. *(5 marks)* Use the Trapezoidal Rule with $n = 4$ to approximate the value of the definite integral and compare your answer to the exact value of the definite integral. *(i.e. calculate the definite integral using the Fundamental Theorem of Calculus.)*

$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx \overset{①,②}{=} \int_1^9 \frac{1}{\sqrt{u}}\frac{du}{3} = \frac{1}{3}\left[2\sqrt{u}\right]_1^9 = \frac{2\sqrt{9}}{3} - \frac{2}{3} = \frac{4}{3}$$

① $u = 1 + x^3$
$du = 3x^2 dx$
② $\frac{du}{3} = x^2 dx$
$u(0) = 1 + 0^3 = 1$
$u(2) = 1 + 2^3 = 9$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2.$

$$\approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{2-0}{2(4)}\left[\frac{0^2}{\sqrt{1+0^3}} + \frac{2(\frac{1}{2})^2}{\sqrt{1+(\frac{1}{2})^3}} + \frac{2(1)^2}{\sqrt{1+1^3}} + \frac{2(\frac{3}{2})^2}{\sqrt{1+(\frac{3}{2})^3}} + \frac{2^2}{\sqrt{1+2^3}}\right]$$

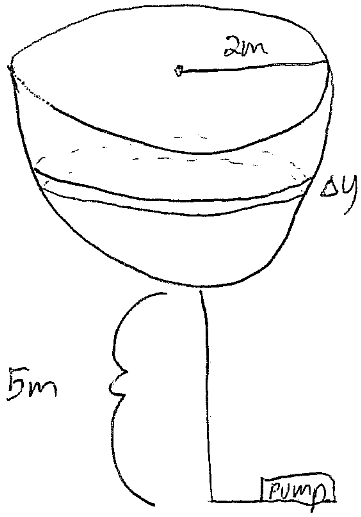
$$= \frac{1}{4}\left[0 + \frac{\frac{1}{2}}{\sqrt{\frac{9}{8}}} + \frac{2}{\sqrt{2}} + \frac{\frac{9}{2}}{\sqrt{\frac{35}{8}}} + \frac{4}{\sqrt{9}}\right]$$

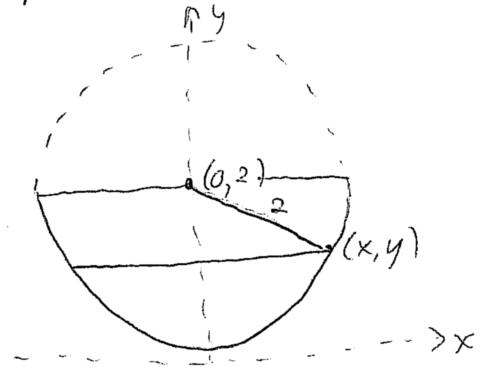
$\doteq 1.343$

Question 8. *(5 marks)* A hemispherical tank with its round part pointing downwards is filled by a pump which is located 5m below the tank. If the fluid that has a density of $\rho = 2000\frac{\text{kg}}{\text{m}^3}$ and the tank is 4m across at the top, how much work is required to fill the tank?

volume of slice: $\Delta V = \pi x^2 \Delta y$

relationship between x and y.





$$x^2 + (y-2)^2 = 2^2$$
$$x^2 = 4 - (y-2)^2$$
$$x^2 = 4 - y^2 + 4y - 4$$
$$x^2 = 4y - y^2$$

$\therefore \Delta V = \pi (4y - y^2)\Delta y$

mass of slice: $\Delta m = \Delta V \rho$

force exerted by slice: $\Delta F = \Delta m g = \Delta V \rho g$

distance from pump to slice: $d = y + 5$

work to move slice: $\Delta W = \Delta F d = \pi \rho g (4y - y^2)(y+5)\Delta y$

$$= \pi \rho g \left[4y^2 - y^3 + 20y - 5y^5\right]$$
$$= \pi \rho g \left[20y - y^2 - y^3\right]$$

$$W = \int_0^2 \pi \rho g \left[20y - y^2 - y^3\right]$$
$$= \pi \rho g \left[10y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2$$
$$= \pi \rho g \left[10(2)^2 - \frac{(2)^3}{3} - \frac{(2)^4}{4}\right]$$
$$= \pi \rho g \left[40 - \frac{8}{3} - \frac{16}{4}\right]$$
$$= \frac{1960000}{3}\pi \text{ N}\cdot\text{m}$$

Question 9. *(5 marks)* Find the 'c' value(s) guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \sec x$ over $[\frac{-\pi}{4}, \frac{\pi}{4}]$.

$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\, dx = \sec c \left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$\Big[\ln|\sec x+\tan x|\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sec c\ \frac{\pi}{2}$$

$$\sec c\left(\frac{\pi}{2}\right) = \ln|\sec\tfrac{\pi}{4}+\tan\tfrac{\pi}{4}| - \ln|\sec\tfrac{-\pi}{4}+\tan\tfrac{-\pi}{4}|$$

$$\sec c\left(\frac{\pi}{2}\right) = \ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|$$

$$\frac{1}{\cos c}\left(\frac{\pi}{2}\right) = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\cos c = \left(\frac{\pi}{2}\right) \Big/ \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$c \doteq 0.4710$ rad

$\doteq 27.0°$

and the other solution -0.4710 rad or $-27°$

Bonus Question. *(5 marks)*
Prove one of the following statement:

- If $f(x)$ is an odd function then

$$\int_{-a}^{a} f(x)\,dx = 0$$

- The Fundamental Theorem of Calculus. (in class notes)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

$$= \int_a^0 f(-u)\,{-du} + \int_0^a f(x)\,dx$$ using $u=-x$ as a substitution

$$= \int_a^0 -f(u)\,{-du} + \int_0^a f(x)dx$$ since $f(x)$ is odd.

$$= -\int_0^a -f(u)\,{-du} + \int_0^a f(x)dx$$

$$= 0$$