Dawson College: **Calculus II**: 201-NYB-05-S1: Summer 2008

Name: YANN LAMONTAGNE
Student ID:

Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Find the average of the function $f(x) = x\cos x^2$ over the interval $[-\sqrt{\pi}, \sqrt{\pi}]$

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\sqrt{\pi}-(-\sqrt{\pi})}\int_{-\sqrt{\pi}}^{\sqrt{\pi}} x\cos x^2\,dx$$

$$= \frac{1}{2\sqrt{\pi}}\, 0$$

$$= 0$$

Lets show that $f(x)$ is odd

$$f(-x) = -x\cos(-x)^2$$
$$= -x\cos x^2$$
$$= -f(x)$$

Question 2. *(5 marks)* Evaluate the following definite integral:

$$\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx = \int_{-2}^{1} \frac{1}{(x+2)^2+9}\,dx$$

Complete the square:

$$x^2+4x+13$$
$$= x^2+4x+4-4+13$$
$$= (x+2)^2+9$$

① $u = x+2$
② $du = dx$

$u(-2) = -2+2 = 0$
$u(1) = 1+2 = 3$

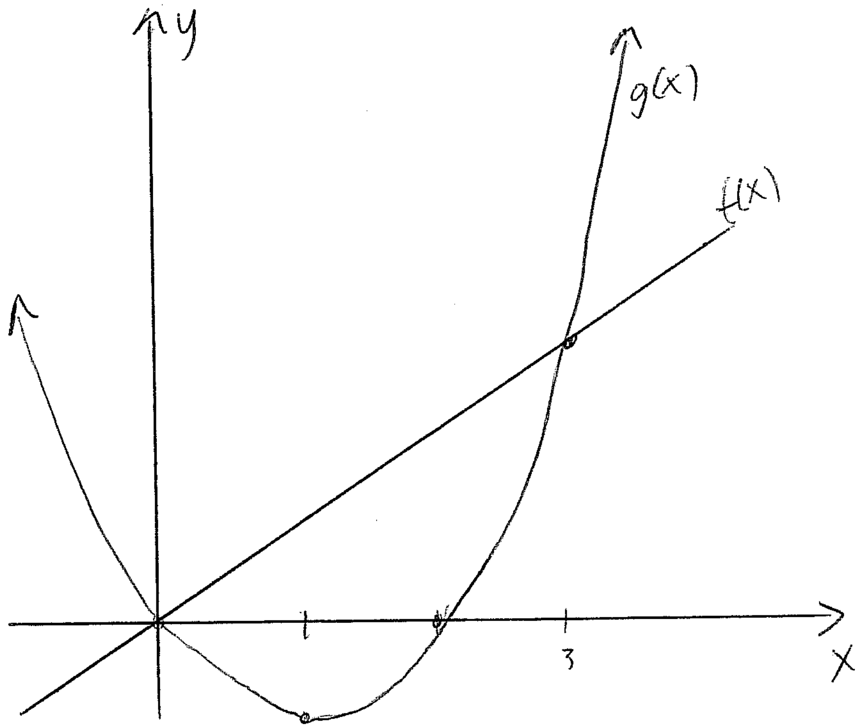
$$\overset{①②}{=} \int_0^3 \frac{1}{u^2+9}\,du$$

$$= \left[\frac{1}{3}\arctan\frac{u}{3}\right]_0^3$$

$$= \frac{1}{3}\arctan 1 - \arctan 0$$

$$= \frac{\pi}{12}$$

Question 3. *(2 marks for sketch and 3 marks for area)* Sketch the graph of the two following algebraic functions $f(x) =$ X and $g(x) =$ X^2-2X and find the area bounded by the two functions.



Lets find the intersection between the two curves

$f(x) = g(x)$

$X = X^2 - 2X$

$0 = X^2 - 3X$

$0 = X(X-3)$

$\therefore$ intersection at $x=0,\ x=3$

$$\text{Area} = \int_0^3 f(x) - g(x)\,dx$$

$$= \int_0^3 x - x^2 + 2x\,dx$$

$$= \int_0^3 -x^2 + 3x\,dx$$

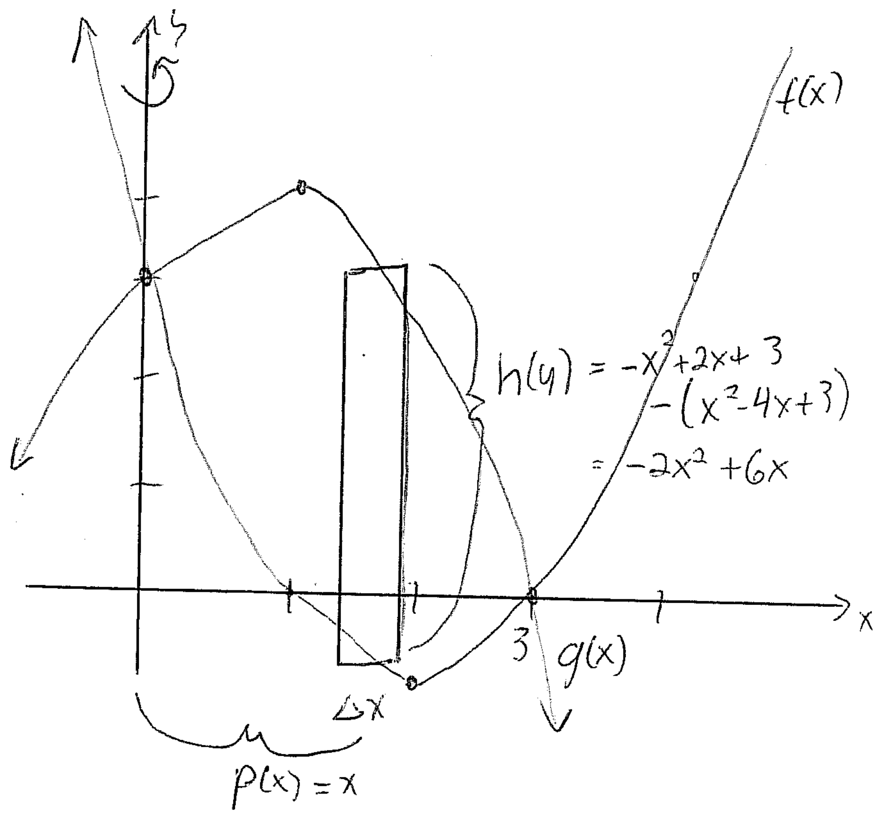
$$= \left[\frac{-x^3}{3} + \frac{3x^2}{2} \right]_0^3$$

$$= -\frac{3^3}{3} + \frac{3(3)^2}{2}$$

$$= -9 + \frac{27}{2} = \frac{9}{2}$$

Question 4. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the function $f(x) = x^2 - 4x + 3$; revolved about the y-axis.

$g(x) = -x^2 + 2x + 3$



Lets find the intersection between the two curves

$$f(x) = g(x)$$

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$0 = 2x^2 - 6x$$

$$0 = 2x(x - 3)$$

$\therefore$ intersection at $x = 0$ and $x = 3$

representative element: $\Delta V = 2\pi p(x) h(x) \Delta x$

$$= 2\pi x(-2x^2 + 6x)\Delta x$$

$$= 2\pi(-2x^3 + 6x^2)\Delta x$$

$$V = \int_0^3 2\pi(-2x^3 + 6x^2)\,dx$$

$$= 2\pi\left[\frac{-2x^4}{4} + \frac{6x^3}{3}\right]_0^3$$

$$= 2\pi\left[\frac{-2(3)^4}{4} + \frac{6(3)^3}{3}\right]$$

$$= 27\pi$$

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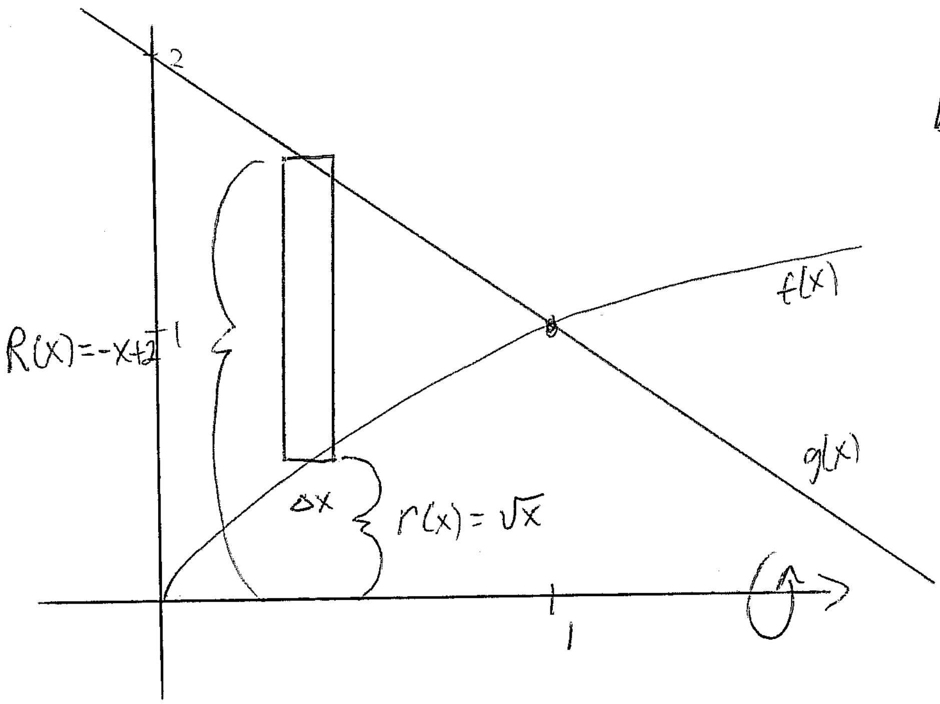
Question 5. *(5 marks)* Find the volume of the solid of revolution generated by the bounded region of the functions $f(x)=\sqrt{x}$, $g(x)=-x+2$ and the relation $x=0$ revolved about the x-axis.



Lets find the intersection between the two curves.

$$f(x) = g(x)$$
$$\sqrt{x} = -x+2$$
$$x = (-x+2)^2 \text{ sbs}$$
$$x = x^2 - 4x + 4$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-4)(x-1)$$

$\therefore$ only valid intersection $x = 1$

representative element:

$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$
$$= \pi\left[(-x+2)^2 - (\sqrt{x})^2\right]\Delta x$$
$$= \pi\left[x^2 - 4x + 4 - x\right]\Delta x$$
$$= \pi\left[x^2 - 5x + 4\right]\Delta x$$

$$V = \int_0^1 \pi\left[x^2 - 5x + 4\right]dx$$
$$= \pi\int_0^1 x^2 - 5x + 4\,dx$$
$$= \pi\left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^1$$
$$= \pi\left[\frac{1}{3} - \frac{5}{2} + 4\right]$$
$$= \frac{11\pi}{6}$$

Question 6. *(5 marks)* Find the arc length of the graph of the function $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ over the interval $[1,2]$.

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

First we take the derivative $f'(x) = \frac{4x^3}{8} - \frac{2}{4x^3}$

$$= \frac{x^3}{2} - \frac{1}{2x^3}$$

Lets simplity the radicand.

$$1 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2 = 1 + \frac{x^6}{4} + \frac{1}{4x^6} - \frac{1}{2}$$

$$= \frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}$$

$$= \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2$$

$$\therefore \quad S = \int_1^2 \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2}\, dx$$

$$= \int_1^2 \frac{x^3}{2} + \frac{1}{2x^3}\, dx$$

$$= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2$$

$$= \frac{2^4}{8} - \frac{1}{4(2)^2} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{16}{8} - \frac{1}{16} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{33}{16}$$

Question 7. *(5 marks)* Use the Simpson's Rule with $n = 4$ to approximate the value of the definite integral and compare your answer to the exact value of the definite integral. *(i.e. calculate the definite integral using the Fundamental Theorem of Calculus.)*

$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx = \int_1^9 \frac{1}{\sqrt{u}} \frac{du}{3} = \frac{1}{3}\int_1^9 \frac{du}{\sqrt{u}} = \frac{1}{3}\Big[2\sqrt{u}\Big]_1^9$$

$u = 1 + x^3$

$du = 3x^2 dx$

$u(0) = 1 + 0^3 = 1$

$u(2) = 1 + 2^3 = 9$

$\frac{du}{3} \; x^2 dx$

$$= \frac{2}{3}\sqrt{9} - \frac{2\sqrt{1}}{3}$$

$$= \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$$

$$\approx \frac{b-a}{3n}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\Big]$$

$\Delta x = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}$

$$= \frac{2-0}{3(4)}\left[\frac{0}{\sqrt{1+0^2}}dx + \frac{4(\frac{1}{2})^2}{\sqrt{1+(\frac{1}{2})^3}} + \frac{2(1)^2}{\sqrt{1+(1)^3}} + \frac{4(\frac{3}{2})^2}{\sqrt{1+(\frac{3}{2})^3}} + \frac{2^2}{\sqrt{1+2^2}}\right]$$

$x_0 = 0$

$x_1 = 1/2$

$x_2 = 1$

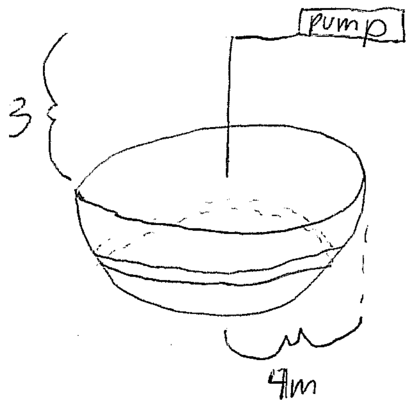
$x_3 = 3/2$

$x_4 = 2$

$$= \frac{2}{12}\left[0 + \frac{1}{\sqrt{9/8}} + \frac{2}{\sqrt{2}} + \frac{9}{\sqrt{35/8}} + \frac{4}{\sqrt{9}}\right]$$

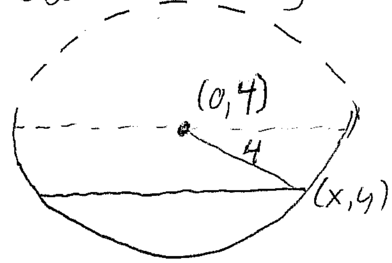
$$\doteq 1.332$$

Question 8. *(5 marks)* A hemispherical tank with its round part pointing downwards is emptied by a pump which is located 3m above the tank. If the fluid that has a density of $\rho = 2000\frac{\text{kg}}{\text{m}^3}$ and the tank is 8m across at the top, how much work is required to empty the tank?



Volume of slice: $\Delta V = \pi x^2 \Delta y$

relationship between x and y:



$$x^2 + (y-4)^2 = 4^2$$
$$x^2 = 16 - (y-4)^2$$
$$x^2 = -y^2 + 8y$$

$\therefore \Delta V = \pi(8y - y^2)\Delta y$

mass of slice: $\Delta m = \rho \Delta V$

force exerted by slice: $\Delta F = \Delta m g = \rho g \Delta V$

distance from slice to pump: $d = 4 + 3 - y = 7 - y$

work to move slice: $\Delta W = \Delta F d = \rho g \Delta V (7-y) = \rho g \pi (8y - y^2)(7-y)\Delta y$

$$= \rho g \pi (56y - 7y^2 - 8y^2 + y^3)\Delta y$$
$$= \rho g \pi (56y - 15y^2 + y^3)\Delta y$$

$$\therefore W = \int_0^4 \rho g \pi (56y - 15y^2 + y^3)\,dy$$
$$= \rho g \pi \int_0^4 (56y - 15y^2 + y^3)\,dy$$
$$= \rho g \pi \left[56\frac{y^2}{2} - \frac{15y^3}{3} + \frac{y^4}{4}\right]_0^4$$
$$= \rho g \pi \left[56\frac{(4)^2}{2} - \frac{15(4)^3}{3} + \frac{4^4}{4}\right]$$
$$= \rho g \pi [192]$$
$$= 3\,763\,200\pi \text{ N}\cdot\text{m}$$

Question 9. *(5 marks)* Find the 'c' value(s) guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \sec x$ over $[\frac{-\pi}{3}, \frac{\pi}{3}]$.

$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\int_{-\pi/3}^{\pi/3} \sec x\, dx = \sec c\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right)$$

$$\Big[\ln|\sec x + \tan x|\Big]_{-\pi/3}^{\pi/3} = \sec c\left(\frac{2\pi}{3}\right)$$

$$\left(\frac{2\pi}{3}\right)\sec c = \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right|$$

$$\left(\frac{2\pi}{3}\right)\sec c = \ln|2+\sqrt{3}| - \ln|2-\sqrt{3}|$$

$$\left(\frac{2\pi}{3}\right)\sec c = \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$\left(\frac{2\pi}{3}\right)\frac{1}{\cos c} = \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$\cos c = \left(\frac{2\pi}{3}\right) \Big/ \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

$$c \doteq 0.6515 \text{ rad}$$
$$\doteq 37.3^\circ$$

the other solutions -37.3° and -0.6515 rad

Bonus Question. *(5 marks)*
Prove one of the following statement:

- If $f(x)$ is an odd function then

$$\int_{-a}^{a} f(x)\,dx = 0$$

- The Fundamental Theorem of Calculus. (in class notes)

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

substitution for first integral $u = -x$

$$= \int_a^0 f(-u)\,{-du} + \int_0^a f(x)dx$$

$$= \int_a^0 -f(u)(-du) + \int_0^a f(x)dx$$

since $f(x)$ is odd

$$= -\int_0^a f(u)du + \int_0^a f(x)dx$$

$$= 0$$