Dawson College: **Calculus II**: 201-NYB-05-S1: Summer 2008

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Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Integrate the following indefinite integral:

$$\int_0^1 \arctan x\,dx = [uv]_0^1 - \int_0^1 v\,du$$

$u = \arctan x \quad du = \frac{dx}{1+x^2}$

$v = x \quad dv = dx$

$$= [x\arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$$

$$= \arctan 1 - \left[\frac{1}{2}\ln|1+x^2|\right]_0^1$$

$$= \pi/4 - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1$$

$$= \pi/4 - \tfrac{1}{2}\ln 2$$

Question 2. *(5 marks)* Integrate the following indefinite integral:

$$\int \tan^3(4x)sec^3(4x)\,dx = \int \tan^2(4x)\sec^2(4x)\tan(4x)\sec(4x)dx$$

① $u = \sec 4x$

$du = \tan 4x \sec 4x (4) dx$

② $\frac{du}{4} = \tan 4x \sec 4x\, dx$

$$= \int (-1+\sec^2(4x))\sec^2(4x)\tan(4x)\sec(4x)dx$$

$$\overset{①,②}{=} \int (-1+u^2)u^2 \frac{du}{4}$$

$$= \frac{1}{4}\int -u^2 + u^4\,du$$

$$= \frac{1}{4}\left[-\frac{u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= -\frac{\sec^3 4x}{12} + \frac{\sec^5 4x}{20} + C$$

Question 3. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx$$

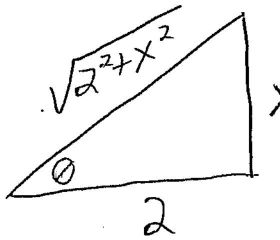
① $x = 2\tan\theta$

② $dx = 2\sec^2\theta\,d\theta$

③ $u = \sec\theta$

④ $du = \tan\theta\sec\theta\,d\theta$

$$\frac{x}{2} = \tan\theta$$



$$\sec\theta = \frac{h}{a} = \frac{\sqrt{4+x^2}}{2}$$

$$\overset{①②}{=} \int \frac{(2\tan\theta)^3\, 2\sec^2\theta\,d\theta}{\sqrt{4+(2\tan\theta)^2}}$$

$$= \int \frac{8\tan^3\theta\; 2\sec^2\theta\,d\theta}{\sqrt{4(1+\tan^2\theta)}}$$

$$= \int \frac{16\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{16\tan^3\theta\sec^{\cancel{2}}\theta\,d\theta}{2\cancel{\sec\theta}}$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int \tan^2\theta\;\tan\theta\sec\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

$$\overset{③,④}{=} 8\int (u^2-1)\,du$$

$$= 8\left(\frac{u^3}{3} - u\right) + c$$

$$= \frac{8\sec^3\theta}{3} - 8\sec\theta + c$$

$$= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\,\frac{\sqrt{4+x^2}}{2} + C$$

Question 4. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{x+1}{x(x^2+1)}\,dx$$ Lets use partial fractions.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x+1 = A(x^2+1) + (Bx+C)x$$

Let $x=0$

$\Rightarrow$ $1 = A(0^2+1) + (B(0)+C)0$

$1 = A$

Let $x=-1$

$\Rightarrow$ $-1+1 = 1((-1)^2+1) + (-B+C)(-1)$

$0 = 2 + B - C$

① $C - B = 2$

Let $x=1$

$\Rightarrow$ $1+1 = 1(1^2+1) + (B+C)(1)$

② $0 = B + C$

Using ① and ②

$2C = 2$

$C = 1$

$\Rightarrow$ $B = -1$

$$\therefore \int \frac{x+1}{x(x^2+1)}\,dx = \int \frac{1}{x} + \frac{-x+1}{x^2+1}\,dx$$

$$= \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \arctan|x| + C$$

Question 5. *(5 marks)* Evaluate the limit, using L'Hôpital's Rule if necessary.

$$\lim_{x \to 0^+} (e^x + x)^{2/x}$$

Let $y = \lim_{x \to 0^+} (e^x + x)^{2/x}$

$$\ln y = \ln \lim_{x \to 0^+} (e^x + x)^{2/x}$$

$$\ln y = \lim_{x \to 0^+} \ln (e^x + x)^{2/x}$$

$$\ln y = \lim_{x \to 0^+} \frac{2 \ln (e^x + x)}{x}$$

has indeterminate form $\frac{0}{0}$ $\therefore$ we use L'Hôpital's Rule

$$\ln y = \lim_{x \to 0^+} \frac{2\left(\frac{e^x + 1}{e^x + x}\right)}{1}$$

$$\ln y = \lim_{x \to 0^+} 2\left(\frac{e^x + 1}{e^x + \cancel{x}}\right)$$

$$\ln y = 4$$

$$y = e^4$$

Question 6. *(5 marks)* Solve the following improper integral:

$$\int_0^\infty xe^{-x}\,dx$$

$u = x \quad du = dx$

$v = -e^{-x} \quad dv = e^{-x}dx$

$$= \lim_{b\to\infty} \int_0^b xe^{-x}dx$$

$$= \lim_{b\to\infty} \left[\left[uv\right]_0^b - \int_0^b v\,du \right]$$

$$= \lim_{b\to\infty} \left[\left[-xe^{-x}\right]_0^b + \int_0^b e^{-x}dx \right]$$

$$= \lim_{b\to\infty} \left[-be^{-b} - \left[e^{-x}\right]_0^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{-b}{e^b} - e^{-b} + e^0 \right]$$

$$= \lim_{b\to\infty} \left[\frac{-b}{e^b} - \cancel{\frac{1}{e^b}}^{\,0} + 1 \right]$$

$$= 1 + \lim_{b\to\infty} \frac{-b}{e^b}$$

IF $\frac{-\infty}{\infty}$ $\therefore$ we use L'Hopital's rule.

$$= 1 + \lim_{b\to\infty} \cancel{\frac{-1}{e^b}}^{\,0}$$

$$= 1$$

Question 7. *(5 marks)* Solve the following improper integral:

$$\int_0^6 \frac{4}{(6-x)^2}\,dx$$

$$= \lim_{b \to 6^-} \int_0^b \frac{4}{(6-x)^2}\,dx$$

① $u = 6 - x$

$du = -dx$

② $-du = dx$

$u(0) = 6$

$u(b) = 6 - b$

$$\overset{①,②}{=} \lim_{b \to 6^-} \int_6^{6-b} \frac{-4\,du}{u^2}$$

$$= \lim_{b \to 6^-} \left[\frac{4}{u}\right]_6^{6-b}$$

$$= \lim_{b \to 6^-} \left[\frac{4}{6-b}^{\nearrow \infty} - \frac{4}{6}\right]$$

$\therefore$ the improper integral diverges.

Question 8. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{1}{\cos x - 1}\,dx$$

$$= \int \frac{1}{\cos x - 1}\left(\frac{\cos x + 1}{\cos x + 1}\right) dx$$

$$= \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx$$

$$= -\int \frac{\cos x + 1}{\sin^2 x}\,dx$$

$$= -\int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx$$

$$= + \frac{1}{\sin x} + \int \csc^2 x\,dx$$

$$= \frac{1}{\sin x} - \cot x + C$$

Question 9 *(5 marks)* Determine the convergence or divergence of the sequence with the given n^{th} term. If the sequence converges find its limit.

$b_n = ne^{-n/2}$

$$\lim_{n\to\infty} ne^{-n/2} = \lim_{n\to\infty} \frac{n}{e^{n/2}}$$

IF $\frac{\infty}{\infty}$ $\therefore$ we use L'Hopital's rule

$$= \lim_{n\to\infty} \frac{2}{e^{n/2}}$$

$$= 0$$

Bonus Question. *(3 marks)*

$$\int \frac{e^x}{e^{2x}(e^x+1)}\,dx$$

$$\int \frac{e^x}{(e^x)^2(e^x+1)}\,dx$$

let ① $u = e^x$
② $du = e^x dx$

$$\overset{①,②}{=} \int \frac{du}{u^2(u+1)}$$

lets now use partial fractions

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$1 = Au(u+1) + B(u+1) + Cu^2$$

Let $u=0$

$\Rightarrow 1 = A(0)(0+1) + B(0+1) + C(0)^2$

$1 = B$

Let $u=-1$

$1 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$

$1 = C$

Let $u=1$

$1 = A(1)(1+1) + B(1+1) + C(1)^2$

$1 = 2A + 2B + C$

$1 = 2A + 2 + 1$

$-1 = A$

$$\therefore \int \frac{1}{u^2(u+1)}\,du = \int \frac{-1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\,du$$

$$= -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

$$= -\ln e^x - \frac{1}{e^x} + \ln|e^x+1| + C$$

$$= -x - \frac{1}{e^x} + \ln|e^x+1| + C$$