Dawson College: **Calculus II**: 201-NYB-05-S1: Summer 2008

Name: Yann Lamontagne
Student ID: ____________

Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Integrate the following indefinite integral:

$$\int_0^1 \arcsin x \, dx$$

$u = \arcsin x \qquad du = \dfrac{dx}{\sqrt{1-x^2}}$

$v = x \qquad dv = dx$

$$= \Big[uv\Big]_0^1 - \int_0^1 v\,du$$

$$= \Big[x\arcsin x\Big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin 1 + \Big[\sqrt{1-x^2}\Big]_0^1$$

$$= \frac{\pi}{2} - 1$$

Question 2. *(5 marks)* Integrate the following indefinite integral:

$$\int \tan^4(3x)\sec^4(3x)\,dx$$

① $u = \tan(3x)$

$du = \sec^2(3x)\,3dx$

② $\dfrac{du}{3} = \sec^2(3x)dx$

$$= \int \tan^4(3x)\sec^2(3x)\sec^2(3x)dx$$

$$= \int \tan^4(3x)(\tan^2(3x)+1)\sec^2(3x)\,dx$$

$$\overset{①,②}{=} \frac{1}{3}\int u^4(u^2+1)du$$

$$= \frac{1}{3}\int u^6 + u^4\,du$$

$$= \frac{1}{3}\left[\frac{u^7}{7} + \frac{u^5}{5}\right] + C$$

$$= \frac{\tan^7 3x}{21} + \frac{\tan^5 3x}{15} + C$$

Question 3. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{x^3}{\sqrt{x^2-9}}\,dx$$

① $x = 3\sec\theta$

② $dx = 3\sec\theta\tan\theta\,d\theta$

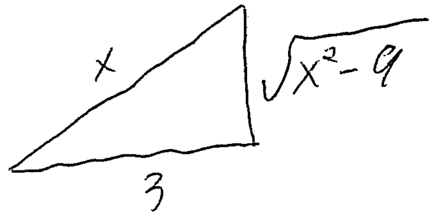
$$\overset{①,②}{=} \int \frac{(3\sec\theta)^3\, 3\sec\theta\tan\theta\,d\theta}{\sqrt{(3\sec\theta)^2-9}}$$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= 81\int \frac{\sec^4\theta\tan\theta\,d\theta}{\sqrt{9(\sec^2\theta-1)}}$$

① $\frac{h}{a} = \frac{x}{3} = \sec\theta$



$\tan\theta = \frac{\sqrt{x^2-9}}{3}$

$$= 81\int \frac{\sec^4\theta\tan\theta\,d\theta}{\sqrt{9\tan^2\theta}}$$

$$= 27\cancel{81}\int \frac{\sec^4\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{3}\cancel{\tan\theta}}$$

$$= 27\int \sec^4\theta\,d\theta$$

$$= 27\int \sec^2\theta\sec^2\theta\,d\theta$$

$$= 27\int (\tan^2\theta+1)\sec^2 d\theta$$

$$= 27\int u^2+1\,du$$

$$= 27\left[\frac{u^3}{3}+u\right] + C$$

$$= \overset{9}{\cancel{27}}\frac{\tan^3\theta}{\cancel{3}} + 27\tan\theta + C$$

$$= 9\cdot\left(\frac{\sqrt{x^2-9}}{3}\right)^3 + 27\frac{\sqrt{x^2-9}}{3} + C$$

$$= \frac{(\sqrt{x^2-9})^3}{3} + 9\sqrt{x^2-9} + C$$

Question 4. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{x-1}{(x+1)(x^2+2)}\,dx$$ Lets use partial fractions.

$$\frac{x-1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

$$x-1 = A(x^2+2) + (Bx+C)(x+1)$$

Let $x=-1$

$$-1-1 = A((-1)^2+2) + (B(-1)+C)(-1+1)$$

$$-2 = 3A$$

$$\frac{-2}{3} = A$$

Let $x=0$

$$0-1 = A(0^2+2) + (B(0)+C)(0+1)$$

$$-1 = 2A + C$$

$$-1 = 2\left(\frac{-2}{3}\right) + C$$

$$\frac{1}{3} = C$$

Let $x=1$

$$1-1 = A(1^2+2) + (B(1)+C)(1+1)$$

$$0 = 3A + 2B + C$$

$$0 = 3\left(\frac{-2}{3}\right) + 2B + \frac{2}{3}$$

$$\frac{2}{3} = B$$

$$\therefore \int \frac{x-1}{(x+1)(x^2+2)}\,dx = \int \frac{-2/3}{x+1} + \frac{\frac{2}{3}x + \frac{1}{3}}{x^2+2}\,dx$$

$$= -\frac{2}{3}\ln|x+1| + \frac{2}{3}\int \frac{x}{x^2+2}\,dx + \frac{1}{3}\int \frac{1}{x^2+2}\,dx$$

$$= -\frac{2}{3}\ln|x+1| + \frac{1}{3}\ln|x^2+2| + \frac{1}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C$$

Question 5. *(5 marks)* Evaluate the limit, using L'Hôpital's Rule if necessary.

$$\lim_{x\to 4^+} (3(x-4))^{x-4}$$

let $y = \lim_{x\to 4^+} (3(x-4))^{x-4}$

$$\ln y = \ln \lim_{x\to 4^+} (3(x-4))^{x-4}$$

$$\ln y = \lim_{x\to 4^+} \ln (3(x-4))^{x-4}$$

$$\ln y = \lim_{x\to 4^+} (x-4) \ln (3(x-4))$$

$$\ln y = \lim_{x\to 4^+} \frac{\ln(3(x-4))}{1/(x-4)}$$

has indeterminate form $\frac{-\infty}{\infty}$ $\therefore$ we use L'Hopital's rule

$$\ln y = \lim_{x\to 4^+} \frac{\frac{3x}{3(x-4)}}{\frac{-1}{(x-4)^2}}$$

$$\ln y = \lim_{x\to 4^+} \frac{-(x-4)^{2}}{\cancel{(x-4)}}$$

$$\ln y = 0$$

$$y = 1$$

Question 6. *(5 marks)* Solve the following improper integral:

$$\int_1^\infty \frac{3}{(x+2)^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{3}{(x+2)^2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{-3}{(x+2)}\right]_1^b$$

$$= \lim_{b\to\infty} \cancel{\frac{-3}{(b+2)}}^{\,0} + \frac{3}{(1+2)}$$

$$= 1$$

Question 7 *(5 marks)* Solve the following improper integral:

$$\int_0^1 x \ln x \, dx$$

$u = \ln x \qquad du = \frac{1}{x} dx$

$v = \frac{x^2}{2} \qquad dv = x\,dx$

$$= \lim_{a \to 0^+} \int_a^1 x \ln x \, dx$$

$$= \lim_{a \to 0^+} \left[\left[uv \right]_a^1 - \int_a^1 v \, du \right]$$

$$= \lim_{a \to 0^+} \left[\left[\frac{x^2}{2} \ln x \right]_a^1 - \int_a^1 \frac{x^2}{2x} dx \right]$$

$$= \lim_{a \to 0^+} \left[-\frac{a^2}{2} \ln a - \left[\frac{x^2}{4} \right]_a^1 \right]$$

$$= \lim_{a \to 0^+} \left[\frac{-a^2 \ln a}{2} - \frac{1}{4} + \frac{a^2}{4} \xrightarrow{} 0 \right]$$

$$= -\frac{1}{4} - \lim_{a \to 0^+} \frac{a^2 \ln a}{2}$$

$$= -\frac{1}{4} - \frac{1}{2} \lim_{a \to 0^+} \frac{\ln a}{1/a^2}$$ has IF $\frac{-\infty}{\infty}$ $\therefore$ we use l'Hopital's rule

$$= -\frac{1}{4} - \frac{1}{2} \lim_{a \to 0^+} \frac{-a^{\cancel{3}\,2}}{\cancel{2a}} \to 0$$

$$= \frac{-1}{4}$$

Question 8. *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{1}{\sec x - 1}\, dx$$

$$= \int \frac{1}{\sec x - 1} \left(\frac{\sec x + 1}{\sec x + 1} \right) dx$$

$$= \int \frac{\sec x + 1}{\sec^2 x - 1}\, dx$$

$$= \int \frac{\sec x + 1}{\tan^2 x}\, dx$$

$$= \int \frac{\sec x}{\tan^2 x}\, dx + \int \frac{1}{\tan^2 x}\, dx$$

$$= \int \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}}\, dx + \int \cot^2 x\, dx$$

$$= \int \frac{\cos^2 x}{\cancel{\cos x} \sin^2 x}\, dx + \int \csc^2 x - 1\, dx$$

$$= \int \frac{\cos x}{\sin^2 x}\, dx - \cot x - x + C$$

$$= \frac{-1}{\sin x} - \cot x - x + C$$

Question 9. *(5 marks)* Determine the convergence or divergence of the sequence with the given n^{th} term. If the sequence converges find its limit.

$$b_n = \frac{\ln\sqrt{n}}{n}$$

$$\lim_{n\to\infty} \frac{\ln\sqrt{n}}{n} = \lim_{n\to\infty} \frac{\frac{1}{2}\ln n}{n}$$

IF $\frac{\infty}{\infty}$ $\therefore$ use L'Hopital's Rule

$$= \frac{1}{2}\lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$= 0$$

Bonus Question. *(3 marks)*

$$\int \frac{e^x}{e^{2x}(e^x+1)}\,dx$$

$$\int \frac{e^x}{(e^x)^2(e^x+1)}dx$$

let ① $u = e^x$
② $du = e^x dx$

①,② $\int \frac{du}{u^2(u+1)}$ lets now use partial fractions

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$1 = Au(u+1) + B(u+1) + Cu^2$$

Let $u=0$

$\Rightarrow 1 = A(0)(0+1) + B(0+1) + C(0)^2$

$1 = B$

Let $u=-1$

$1 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$

$1 = C$

Let $u=1$

$1 = A(1)(1+1) + B(1+1) + C(1)^2$

$1 = 2A + 2B + C$

$1 = 2A + 2 + 1$

$-1 = A$

$$\therefore \int \frac{1}{u^2(u+1)}du = \int \frac{-1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\,du$$

$$= -\ln|u| - \frac{1}{u} + \ln|u+1| + C$$

$$= -\ln e^x - \frac{1}{e^x} + \ln|e^x+1| + C$$

$$= -x - \frac{1}{e^x} + \ln|e^x+1| + C$$