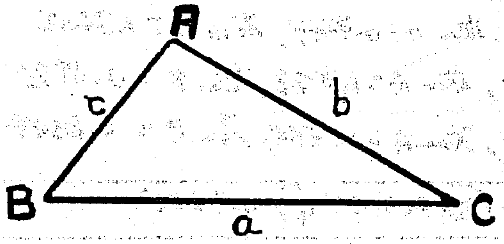
SOLVING OBLIQUE TRIANGLES

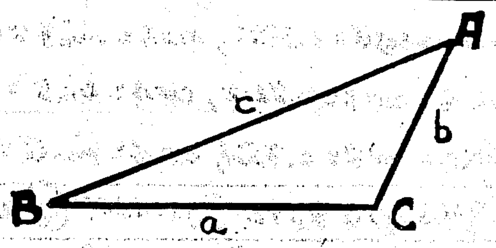
We consider 2 possible **OBLIQUE** (non-right) triangles as shown below:

3 ACUTE ANGLES



or

1 OBTUSE and 2 ACUTE ANGLES



$$\boxed{A + B + C = 180^\circ}$$

THE LAW OF COSINES

SAS | **SSS**

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{and} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{and} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

CASES: **SAS** – given 2 sides and the included angle.

SSS – given 3 sides.

and

THE LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

CASES: **AAS (or ASA)** – given 2 angles and any 1 side.

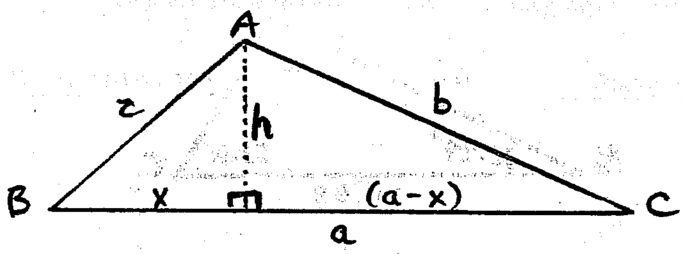
SSA – given 2 sides and the angle opposite one of the sides.

Note: The AAA case (given 3 angles) has no unique solution.

SOLVING OBLIQUE TRIANGLES – EXAMPLES

Derivation of The Law of Cosines:

consider



then

$b^2 = h^2 + (a-x)^2$ where $\cos B = \frac{x}{c}$ and $c^2 = h^2 + x^2$

$\therefore b^2 = h^2 + a^2 - 2ax + h^2$ $\therefore c\cos B = x$ $\therefore c^2 - h^2 = x^2$

substituting $b^2 = \cancel{h^2} + a^2 - 2ac\cos B + c^2 - \cancel{h^2}$

hence $\boxed{b^2 = a^2 + c^2 - 2ac\cos B \quad \text{and} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}}$

Similarily the other statements can be derived.

Note: If $B = 90°$, then $\cos B = \cos 90° = 0$ and the law becomes $c^2 = a^2 + b^2$, the Pythagorean Theorem.

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① (SAS) Solve triangle ABC given $b = 25$, $A = 60°$, and $c = 42$.

Solution: ① consider $a^2 = b^2 + c^2 - 2bc\cos A$, Law of Cosines

$a^2 = 25^2 + 42^2 - 2(25)(42)\cos 60°$

$a^2 = 1339$

$a = \sqrt{1339}$

$\therefore a \simeq 36.59$

② consider $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$, Law of Cosines

$\cos B = \frac{(36.59)^2 + 42^2 - 25^2}{2(36.59)(42)}$

$\cos B = 0.8060$

$\therefore B \simeq 36.29°$, $\cos^{-1}$ in calculator

③ consider $A + B + C = 180°$

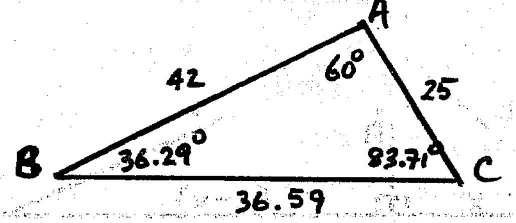
$60° + 36.29° + C = 180°$

$C = 180° - 60° - 36.29°$

$\therefore C \simeq 83.71°$

continued ---

① (continued) hence, the solved triangle ABC is :



Note: For a rough check, consider side-angle proportionality (opposite the smallest angle is the smallest side etc)

② (SAS) Solve triangle ABC given $a = 20$, $C = 33°$, and $b = 10$.

Solution: ① consider $c^2 = a^2 + b^2 - 2ab\cos C$, Law of Cosines

$$c^2 = 20^2 + 10^2 - 2(20)(10)\cos 33°$$

$$c^2 = 164.53$$

$$\therefore c \simeq 12.83$$

② consider $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, Law of Cosines

$$\cos A = \frac{10^2 + (12.83)^2 - 20^2}{2(10)(12.83)}$$

$$\cos A = -0.5276$$

$\therefore A \simeq 121.84°$, $\cos^{-1}$ in calculator

(Recall that $\cos\theta$ is negative when θ is obtuse)

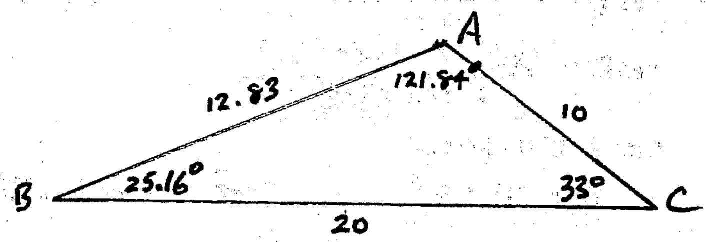
③ consider $A + B + C = 180°$

$$121.84° + B + 33° = 180°$$

$$B = 180° - 121.84° - 33°$$

$$\therefore B \simeq 25.16°$$

hence, the solved triangle is :



SOLVING OBLIQUE TRIANGLES - EXAMPLES

③ (SSS) Solve triangle ABC given $a = 9$, $b = 8$, and $c = 10$.

__Solution:__ ① consider $\cos A = \dfrac{a^2+b^2-c^2}{2ab}$, Law of Cosines

$$\cos A = \frac{9^2+8^2-10^2}{2(9)(8)}$$

$$\cos A = 0.3125$$

$$\therefore\ A \approx 71.79^\circ \quad , \ \cos^{-1} \text{ in calculator}$$

② consider $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, Law of Cosines

$$\cos B = \frac{9^2+10^2-8^2}{2(9)(10)}$$

$$\cos B = 0.6500$$

$$\therefore\ B \approx 49.46^\circ \quad , \ \cos^{-1} \text{ in calculator}$$

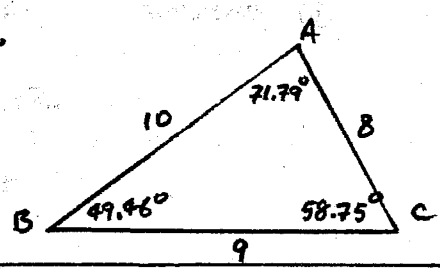
③ consider $A+B+C = 180^\circ$

$$71.79^\circ + 49.46^\circ + C = 180^\circ$$

$$C = 180^\circ - 71.79^\circ - 49.46^\circ$$

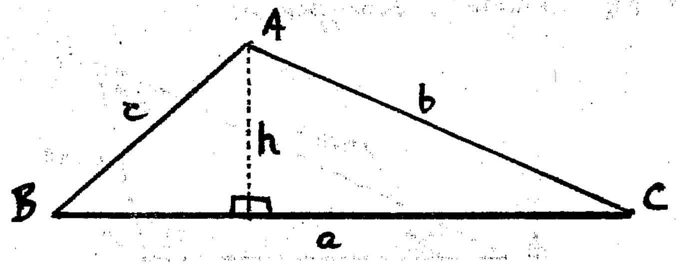
$$\therefore C = 58.75^\circ$$

hence, the solved triangle is:



Derivation of the Law of Sines:

consider



then $\sin B = \dfrac{h}{c}$ and $\sin C = \dfrac{h}{b}$

$\therefore c \sin B = h$ $\qquad$ $\therefore b \sin C = h$

thus $\quad c \sin B = b \sin C$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly $\quad \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

hence $\quad \boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

__Note:__ Since $\sin 90^\circ = 1$, the Law of Sines can be used to solve right triangles.

(4) (AAS) Solve triangle ABC given $A = 35°$, $B = 50°$, and $a = 12$.

Solution: (1) consider $A + B + C = 180°$

$$35° + 50° + C = 180°$$
$$C = 180° - 35° - 50°$$
$$\therefore C = 95°$$

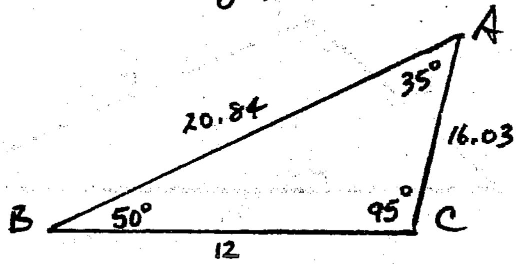
(2) consider $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$, Law of Sines

$$\frac{\sin 35°}{12} = \frac{\sin 50°}{b}$$
$$b = \frac{12 \sin 50°}{\sin 35°}$$
$$\therefore b \approx 16.03$$

(3) consider $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, Law of Sines

$$\frac{\sin 35°}{12} = \frac{\sin 95°}{c}$$
$$c = \frac{12 \sin 95°}{\sin 35°}$$
$$\therefore c \approx 20.84$$

hence, the solved triangle is:



(5) (ASA) Solve triangle ABC if $A = 80^\circ$, $B = 40^\circ$, and $c = 10$.

Solution: (1) consider $A + B + C = 180^\circ$

$$80^\circ + 40^\circ + C = 180^\circ$$
$$C = 180^\circ - 80^\circ - 40^\circ$$
$$\therefore C = 60^\circ$$

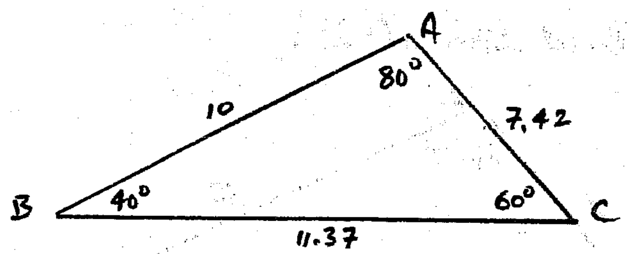
(2) consider $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, Law of Sines

$$\frac{\sin 80^\circ}{a} = \frac{\sin 60^\circ}{10}$$
$$\frac{10 \sin 80^\circ}{\sin 60^\circ} = a$$
$$\therefore a \simeq 11.37$$

(3) consider $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$, Law of Sines

$$\frac{\sin 40^\circ}{b} = \frac{\sin 60^\circ}{10}$$
$$\frac{10 \sin 40^\circ}{\sin 60^\circ} = b$$
$$\therefore b \simeq 7.42$$

hence, the solved triangle is :



⑥ (SSA) Solve triangle ABC given $b = 50$, $c = 30$, and $B = 60^\circ$.

<u>Solution</u>: ① consider $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$, Law of Sines

$$\frac{\sin 60^\circ}{50} = \frac{\sin C}{30}$$

$$\frac{30 \sin 60^\circ}{50} = \sin C$$

then $\sin C = 0.5196$

$$\therefore C \simeq 31.31^\circ$$

(<u>Note</u>: $\sin 148.69^\circ = 0.5196$ also, but C is acute, since $c < b$)

② Consider $A + B + C = 180^\circ$

$$A + 60^\circ + 31.31^\circ = 180^\circ$$

$$A = 180^\circ - 60^\circ - 31.31^\circ$$

$$\therefore A = 88.69^\circ$$

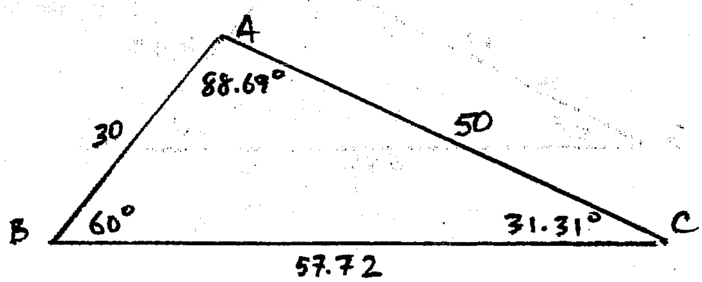
③ consider $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$, Law of Sines

$$\frac{\sin 88.69^\circ}{a} = \frac{\sin 60^\circ}{50}$$

$$\frac{50 \sin 88.69^\circ}{\sin 60^\circ} = a$$

$$\therefore a \simeq 57.72$$

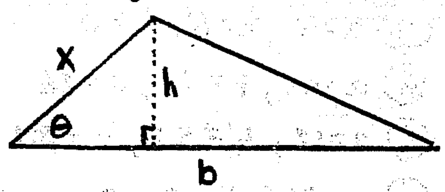
Hence, the solved triangle is:



<u>Note</u>: We exclude solutions of the SSA case when the given angle is not opposite the longest given side. (called the ambiguous case)

A formula for the Area of a Triangle:

Consider a triangle with sides x and b and the included angle θ:



where $\sin\theta = \frac{h}{x}$

$\therefore x\sin\theta = h$

then $\text{Area} = \frac{1}{2}bh$, basic formula

$\text{Area} = \frac{1}{2}b(x\sin\theta)$, substituting

hence $\boxed{\text{area} = \frac{1}{2}bx\sin\theta}$

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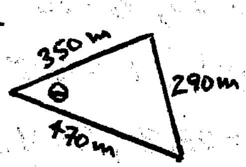
⑦ (Application) Find the area of a triangular lot whose sides measure 350 meters, 290 meters, and 470 meters.

consider



then $\cos\theta = \frac{350^2+470^2-290^2}{2(350)(470)}$, Law of Cosines

$\therefore \theta \simeq 37.99^\circ$

hence, $\text{Area} = \frac{1}{2}(470)(350)\sin 37.99^\circ$, formula above

$\therefore \text{Area} \simeq 50627\ m^2$

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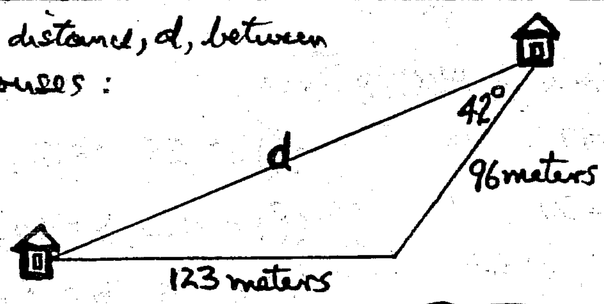
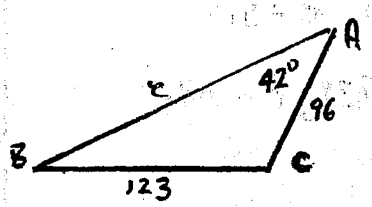
⑧ (Application) Find the distance, d, between the 2 houses:



consider



then $\frac{\sin 42^\circ}{123} = \frac{\sin B}{96}$, Law of Sines

$\therefore B \approx 31.48^\circ$

then $42^\circ + 31.48^\circ + C = 180^\circ$

$\therefore C = 106.52^\circ$

and $\frac{\sin 42^\circ}{123} = \frac{\sin 106.52}{c}$

$\therefore c \simeq 176.23$

hence, $d \simeq 176\ m$

SOLVING OBLIQUE TRIANGLES – EXERCISES

(1) Use the Law of Cosines to solve triangle ABC, given:

(a) $a=4$, $C=60°$, $b=5$
(b) $a=50$, $B=44°$, $c=40$
(c) $b=53.7$, $A=71.17°$, $c=51.7$
(d) $a=5.32$, $C=122°$, $b=3.07$
(e) $a=46.10$, $C=61.4°$, $b=71.43$
(f) $a=1.058$, $B=128.52°$, $c=6.75$
(g) $a=9$, $b=8$, $c=10$
(h) $a=49$, $b=53$, $c=26$
(i) $a=12$, $b=5.2$, $c=8.1$
(j) $a=894$, $b=802.3$, $c=847$
(k) $a=98.41$, $b=73.59$, $c=49.81$

(2) Use the Law of Sines to solve triangle ABC, given:

(a) $A=75°$, $C=50°$, $c=42$
(b) $A=30°$, $B=45°$, $a=4$
(c) $A=120°$, $C=30°$, $c=5$
(d) $B=30°$, $C=135°$, $b=\sqrt{2}$
(e) $B=27.4°$, $C=26.7°$, $a=478.3$
(f) $A=27.6°$, $B=41.3°$, $c=14.92$
(g) $B=38.64°$, $C=91.36°$, $a=119.05$
(h) $A=41.3°$, $B=110.2°$, $b=0.5739$
(i) $b=50$, $c=30$, $B=60°$
(j) $a=\sqrt{3}$, $b=1$, $A=120°$
(k) $a=32.1$, $c=25.3$, $A=78.23°$

(3) Show that the area of triangle ABC is given by:

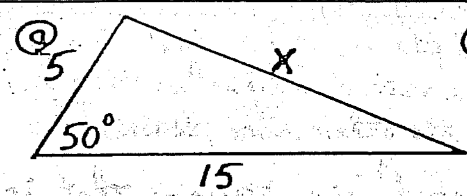
$$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

(4) Find the Area of triangle ABC, given:

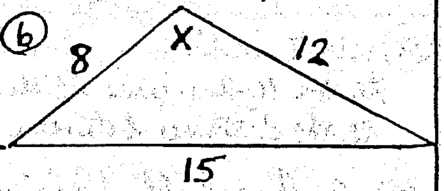
(a) $a=15$, $B=20°$, $c=17$
(b) $b=20$, $A=40°$, $c=30$
(c) $a=14.6$, $C=130.2°$, $b=31.7$
(d) $A=55°$, $B=45°$, $c=12$
(e) $A=102°$, $C=47°$, $b=82$
(f) $B=29°$, $C=46°$, $b=20$
(g) $b=25$, $c=12$, $B=70°$
(h) $a=100$, $b=40$, $A=100°$
(i) $a=5$, $b=7$, $c=10$
(j) $a=57$, $b=85$, $c=110$
(k) $a=976$, $b=728$, $c=543$

(5) Use the Law of Cosines to find x in each triangle to the right:

ⓐ



ⓑ



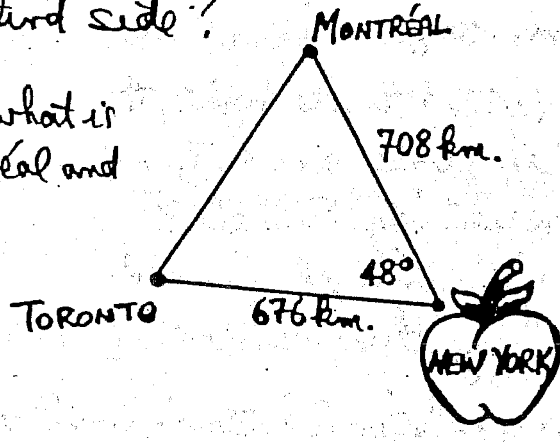
(6) The sides of a parallelogram are 5 feet and 8 feet. One angle is 60° and the other is 120°. What are the lengths of the diagonals?

(7) Two adjacent sides of a parallelogram meet at an angle of 58.50° and have lengths of 18 and 24 feet. Find the lengths of the diagonals.

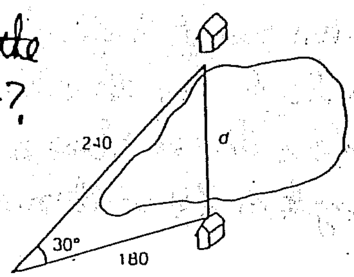
(8) A parallelogram has sides of lengths 45 and 65 meters. If one diagonal is 70 meters long, then find the angles of the parallelogram

(9) At one corner of a triangular field, the angle measures 52.4°. If the sides that meet there are 100 m. and 120 m. long, respectively, then how long is the third side?

(10) In the map to the right, what is the distance between Montréal and Toronto?



(11) In the picture to the right, find the distance, d, between the 2 houses? (distances in meters)



SOLVING OBLIQUE TRIANGLES – EXERCISES (CONTINUED)

(12) While in the air, 1 of 3 planes determines its distance from the other two to be 10 km. and 25 km. respectively when they are 100° apart. Then, what is the distance between the other two planes?

(13) A baseball "diamond" is a square that is 90 feet long on each side. If the pitcher's mound is centered 60.5 ft. from homeplate on the diagonal from home to second base, then:

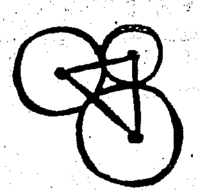
(a) what is the length of this diagonal?

(b) what is the distance between the mound and first base?

(14) A triangle has sides of length 35, 40, and 60 cm. Find the largest angle of the triangle.

(15) A piece of wire 150 cm. long is bent into the shape of a triangle. Find the 3 angles of this triangle, if 2 of its sides are 60 and 50 cm. long.

(16) In the picture to the right, the radii of the circles are 4, 7, and 9 mm. respectively. Find the 3 angles of the triangle formed by joining their centers.

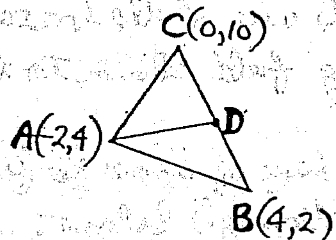


(17) To determine whether 2 interior walls meet at a right angle, carpenters often mark a point 3 ft. from the corner on one wall and a point (at the same height) on the other wall 4 ft. from the corner. If the straight line distance between these points is exactly 5 ft, the walls are square. At what angle do the walls meet if the distance is only 4 feet 10 inches?

SOLVING OBLIQUE TRIANGLES - EXERCISES (CONTINUED)

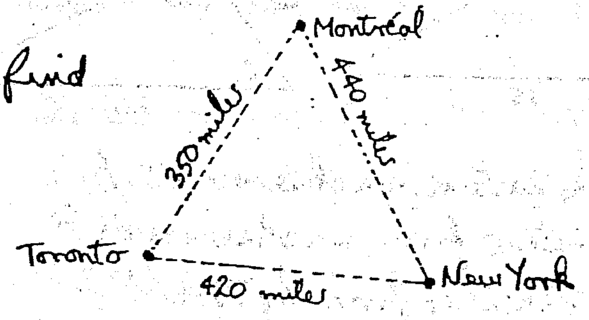
(18) Find angle A in triangle ABC formed by the coordinate points, A(3,4), B(1,5), and C(5,9).

(19) In the picture, find angle ADC if D is the midpoint of CB in the rectangular coordinate system.



(20) Find the area of a triangular lot with sides of 21, 32, and 47 meters.

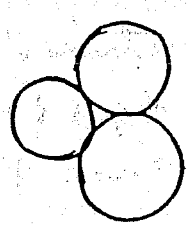
(21) Use the map to the right to find the triangular land area between the 3 cities.



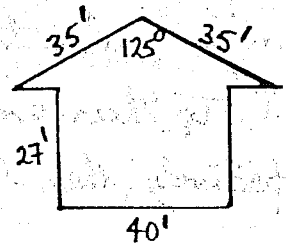
(22) A parallelogram has sides of lengths 10.3 cm. and 23.2 cm., and one of its angles is 54.2°. What is the area of the parallelogram?

(23) A parallelogram has sides of lengths 7.4 and 9.2 inches. If the shortest diagonal is 6.2 inches, what is the area of the parallelogram?

(24) In the picture to the right, the radii of the circles are 5, 7 and 8 mm. respectively. Find the area of the triangle formed by joining their centers.



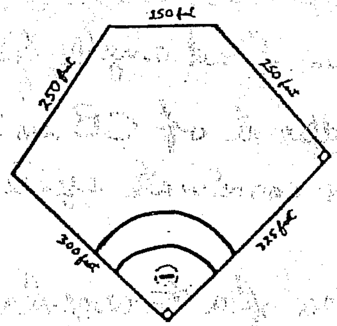
(25) Find the area of the "house end" in the picture to the right.



SOLVING OBLIQUE TRIANGLES – EXERCISES (CONTINUED)

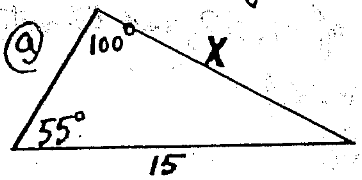
(26) Find the area of the triangle formed by the coordinate points: (0,0), (−2,7), and (3,5)

(27) Find the area of the baseball playing field shown to the right.

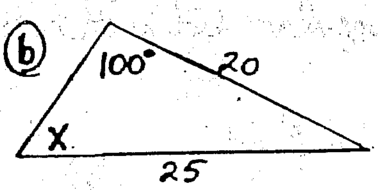


(28) Use the Law of Sines to find X in each triangle below:

(a)



(b)



(29) In the picture, an observer at A is 320 meters from an observer at B. How far is A from the tree?

(30) Two lighthouses A and B along a coast are 8 km. apart and each spot a ship, C, at sea. If angle CAB is 28.25° and angle ABC is 32.33°, then how far is the ship from lighthouse A? If the coast is a straight line, how far is the ship from the coast?

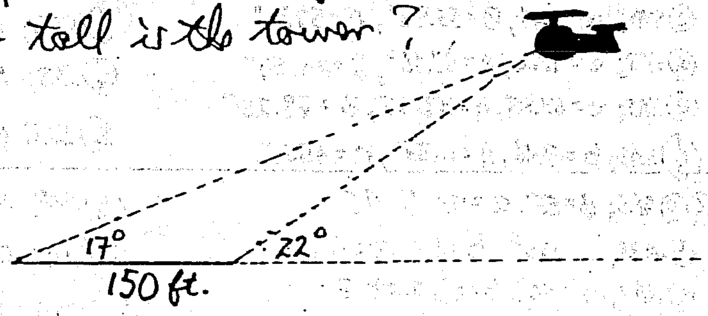
(31) At the same time 2 observers 457.5 meters apart on level ground report an object falling between them with angles of elevation of 32.37° and 43.2°. How high is the object above ground?

(32) On a straight path a tree stands between 2 people who are 400 feet apart. If their angles of elevation to the tree are 25° and 37° respectively, then how high is the tree?

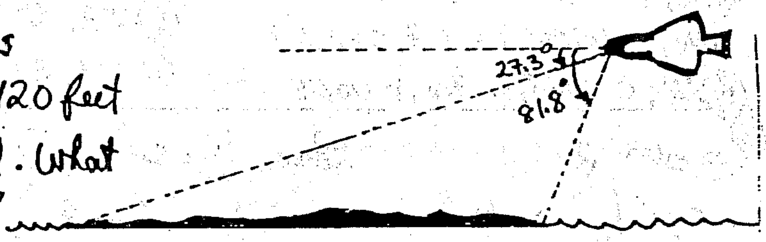
SOLVING OBLIQUE TRIANGLES - EXERCISES (CONTINUED)

(33) From a point 542 feet away from the base of the "Leaning Tower of Pisa", the angle of elevation to its top is 17.9°. If the tower leans away from the point with an angle of 94.5° with the (level) ground, then how tall is the tower?

(34) In the picture, what is the altitude of the helicopter?

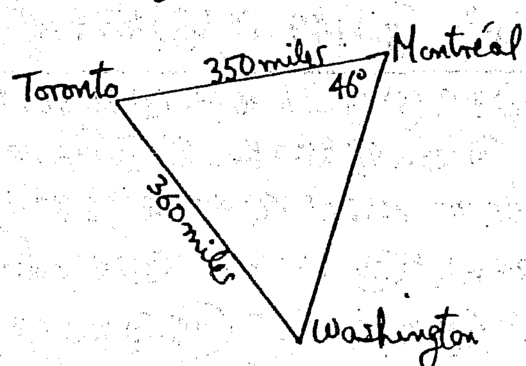


(35) In the picture, the plane is flying at an altitude of 5120 feet as it approaches the island. What is the length of the island?



(36) A 210 foot TV tower stands on the top of a building. From a point on level ground, the angles of elevation to the top and base of the TV tower are 25.2° and 21.1° respectively. How tall is the building

(37) Use the map to estimate the distance between Montréal and Washington.



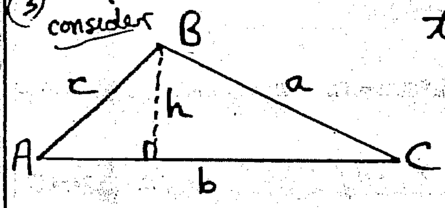
(38) To determine the distance between 2 points A and B on level ground, a surveyor chooses a point C which is 375 m. from A and 530 m. from B. If angle BAC is 49.5°, then find the required distance.

SOLVING OBLIQUE TRIANGLES EXERCISES – ANSWERS

1. (a) SAS; $c=\sqrt{21}$, $A=49°$, $B=71°$
   (b) SAS; $b=35$, $A=83.38°$, $C=52.62°$
   (c) SAS; $a=61.4$, $B=55.93°$, $C=52.90°$
   (d) SAS; $c=7.42$, $A=37.46°$, $B=20.54°$
   (e) SAS; $c=63.83$, $A=39.35°$, $B=79.25°$
   (f) SAS; $b=7.46$, $A=6.37°$, $C=45.11°$
   (g) SSS; $A=58.75°$, $B=49.46°$, $C=71.79°$
   (h) SSS; $A=67°$, $B=84°$, $C=29°$
   (i) SSS; $A=128°$, $B=20°$, $C=32°$
   (j) SSS; $A=65.58°$, $B=54.80°$, $C=59.62°$
   (k) SSS; $A=104.12°$, $B=46.49°$, $C=29.39°$

2. (a) AAS; $B=55°$, $a=53$, $b=45$
   (b) AAS; $C=105°$, $b=5.6$, $c=7.7$
   (c) AAS; $B=30°$, $b=5$, $a=8.7$
   (d) AAS; $A=15°$, $a=0.7$, $c=2$
   (e) AAS; $A=125.90°$, $b=271.7$, $c=265.3$
   (f) AAS; $C=111.10°$, $a=7.41$, $b=10.55$
   (g) AAS; $A=50°$, $b=97.04$, $c=155.37$
   (h) AAS; $C=28.50°$, $a=0.4036$, $c=0.2918$
   (i) SSA; $C=31.31°$, $A=88.69°$, $a=58$
   (j) SSA; $B=30°$, $C=30°$, $c=1$
   (k) SSA; $B=51.27°$, $C=60.50°$, $b=25.6$

3. consider



then $\frac{h}{c}=\sin A$ $\therefore h = c\sin A$ $\Big\}$ $\therefore$ Area $=\frac{1}{2}bh=\frac{1}{2}b(c\sin A)$ $\therefore$ Area $=\frac{1}{2}bc\sin A$

and similarly, Area $=\frac{1}{2}ac\sin B=\frac{1}{2}ab\sin C$

4. (a) $44u^2$ (b) $192.84u^2$ (c) $176.75u^2$ (d) $42u^2$ (e) $4700u^2$
   (f) $286.7u^2$ (g) $149u^2$ (h) $1674u^2$ (i) $16u^2$ (j) $2400u^2$ (k) $195000u^2$

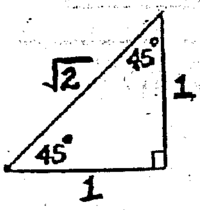
5. (a) 12.4 (b) 95° 6. 7 and 11.36 ft. 7. 21.18 and 36.76 ft. 8. 76.67° and 103.33° 9. 98.8 m.
10. 564 km. 11. 123 m. 12. 28.5 km. 13. (a) 127.28 ft. (b) 63.72 ft. 14. 106.07°
15. 41.41°, 55.77°, 82.82° 16. 43°, 54°, 83° 17. 86° 18. 85.2° 19. 80.0° 20. 280 m.^{2}
21. 68624 mi.^{2} 22. 193 cm.^{2} 23. 46 cm.^{2} 24. 74.8 mm.^{2} 25. 1582 ft.^{2}
26. 15.5 u.^{2} 27. 107211 ft.^{2} 28. (a) 12.5 (b) 51.78° 29. 254 m. 30. 5 and 2 km.
31. 173 m. 32. 115.2 ft. 33. 180 ft. 34. 188.5 ft. 35. 9182 ft. 36. 957 ft.
37. 500 mi. 38. 690 m.

THE SPECIAL ANGLES AND REFERENCE ANGLES

THE SPECIAL ANGLES IN TRIGONOMETRY

The exact values of the trigonometric functions of the special angles are:

THE 45° ANGLE



$\sin 45° = \frac{1}{\sqrt{2}}$

$\csc 45° = \sqrt{2}$

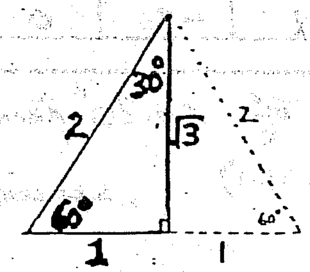
$\cos 45° = \frac{1}{\sqrt{2}}$

$\sec 45° = \sqrt{2}$

$\tan 45° = 1$

$\cot 45° = 1$

THE 60° AND 30° ANGLES



$\sin 60° = \frac{\sqrt{3}}{2} = \cos 30°$

$\csc 60° = \frac{2}{\sqrt{3}} = \sec 30°$

$\cos 60° = \frac{1}{2} = \sin 30°$

$\sec 60° = 2 = \csc 30°$

$\tan 60° = \sqrt{3} = \cot 30°$

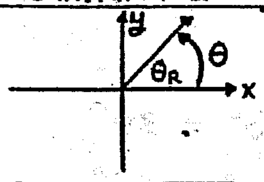
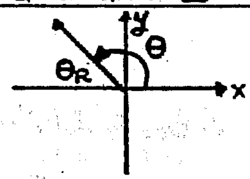
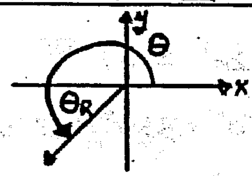
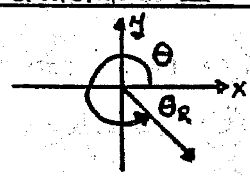
$\cot 60° = \frac{1}{\sqrt{3}} = \tan 30°$

<u>NOTE:</u> The exact values of the trig. functions of quadrantal angles are available in calculators.

REFERENCE ANGLES IN TRIGONOMETRY

THE DEFINITION OF A REFERENCE ANGLE

Every nonquadrantal angle, θ, in standard position has a **REFERENCE ANGLE**, θ_R, defined as the acute angle formed between the terminal side of θ and the x-axis, as shown below by quadrant.

QUADRANT I	QUADRANT II	QUADRANT III	QUADRANT IV
			
$\theta_R = \theta$	$\theta_R = 180° - \theta$	$\theta_R = \theta - 180°$	$\theta_R = 360° - \theta$

To find the <u>exact value</u> of the trigonometric function of an angle θ, where θ_R is a <u>special angle</u>, we use the following relationship.

If **T** is any trigonometric function

$$\boxed{T(\theta) = \pm T(\theta_R)}$$

Determine the $\pm$ sign by the **CAST** rule

<u>NOTE:</u> This relationship is also used to solve trigonometric equations later.

THE SPECIAL ANGLES AND REFERENCE ANGLES – EXAMPLES

For convenience, we include the following table for <u>quadrantal angles</u>.

QUADRANTAL θ's	sin θ	csc θ	cos θ	sec θ	tan θ	cot θ
0°	0	undefined	1	1	0	undefined
90°	1	1	0	undefined	undefined	0
180°	0	undefined	−1	−1	0	undefined
270°	−1	−1	0	undefined	undefined	0

① Find the exact value of $\cos 30^\circ \tan 60^\circ + \sin 45^\circ \cos 90^\circ$.

$\Rightarrow \frac{\sqrt{3}}{2} \cdot \sqrt{3} + \frac{1}{\sqrt{2}} \cdot (-1)$, special angle results

$= \frac{3}{2} - \frac{1}{\sqrt{2}}$

$= \frac{3\sqrt{2} - 2}{2\sqrt{2}}$

$= \frac{6 - 2\sqrt{2}}{4} = \frac{2(3 - \sqrt{2})}{4} = \frac{3 - \sqrt{2}}{2}$, rationalized denominator

② Find the reference angle, θ_R, for each angle:

ⓐ 75° ⓑ 120° ⓒ −150° ⓓ 675°

ⓐ $\theta_R = 75^\circ$

ⓑ $\theta_R = 180^\circ - 120^\circ = 60^\circ$

ⓒ consider $-150^\circ + 360^\circ = 210^\circ$, $\therefore \theta_R = 210^\circ - 180^\circ = 30^\circ$

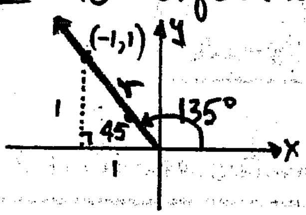
ⓓ consider $675^\circ - 360^\circ = 315^\circ$, $\therefore \theta_R = 360^\circ - 315^\circ = 45^\circ$

③ Find the exact value of $\cos 135^\circ$.

consider $\theta_R = 180^\circ - 135^\circ = 45^\circ$

then $\cos 135^\circ = \pm \cos 45^\circ = -\frac{1}{\sqrt{2}}$, special angle result and CAST

<u>Note</u>: To confirm this answer, consider the following:



then $x = -1$, $y = 1$ and

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

hence, $\cos 135^\circ = \frac{x}{r} = \frac{-1}{\sqrt{2}}$, by definition

THE SPECIAL ANGLES AND REFERENCE ANGLES EXERCISES

(1) Find the exact value of (NOTE: $\sin^2\theta = (\sin\theta)^2$):

(a) $\sin 30°$ (b) $\cos 60°$ (c) $\tan 45°$ (d) $\csc 90°$ (e) $\sec 180°$

(f) $\cot 270°$ (g) $\sin 405°$ (h) $\cos 750°$ (i) $\tan 1140°$

(j) $\sin(-300°)$ (k) $\sec(-1035°)$ (l) $\tan(-690°)$ (m) $\tan(-90°)$

(n) $\sin 60° \cot 30° \tan 45°$ (o) $\sin 45° \cos 45° (\tan 45° + \cot 45°)$

(p) $\cot 60° - \sin 30° \sec 60°$ (q) $\sin 90° \cos 180° + \tan 60° \cot 60°$

(r) $\cos 45° + \cos 180° + \sin 45° + \sin 90° + \sec 45° + \sin 180°$

(s) $\tan 30° + \cos 0° + \cos 30° + \cos 90° + \sec 30° + \sin 270°$

(t) $\tan^2 60° \sec 30° \sin 45°$ (u) $\sin^2 90° \cos^2 60° \sec 30°$

(v) $\csc 90° \sin^2 90° - \cos^2 60°$ (w) $\sin^2 45° + \cos^2 45°$ (x) $\sin^3 3645°$

(2) Find the reference angle for each angle:

(a) $45°$ (b) $135°$ (c) $120°$ (d) $150°$ (e) $225°$ (f) $210°$ (g) $240°$ (h) $315°$

(i) $330°$ (j) $300°$ (k) $50°$ (l) $130°$ (m) $230°$ (n) $310°$ (o) $870°$

(p) $1140°$ (q) $585°$ (r) $-120°$ (s) $-405°$ (t) $-35700°$

(3) Find the exact value of:

(a) $\sin 225°$ (b) $\cos 150°$ (c) $\tan 300°$ (d) $\csc 45°$ (e) $\sec 300°$ (f) $\cot 240°$

(g) $\sin(-855°)$ (h) $\cos(-570°)$ (i) $\cos 135° + \sin 225° + \sec 315°$

(j) $2\sin 150° \cos 330° \cot 60°$ (k) $\tan^2 240° + 2\tan^2 135°$ (l) $2\csc^2 225° - 3\sec^2 150°$

(m) $\tan^2 120° + 4\cos^2 315° + 3\sec^2 210°$ (n) $2\sec^2 180° \cos 0° + 3\sin^3 270° - \csc 90°$

ANSWERS

(1) (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 1 (e) -1 (f) 0 (g) $\frac{1}{\sqrt{2}}$ (h) $\frac{\sqrt{3}}{2}$ (i) $\sqrt{3}$ (j) $\frac{\sqrt{3}}{2}$ (k) $\sqrt{2}$ (l) $\frac{1}{\sqrt{3}}$ (m) undefined
(n) $\frac{3}{2}$ (o) 1 (p) $\frac{1}{\sqrt{3}}-1$ (q) 0 (r) $2\sqrt{2}$ (s) $\frac{3\sqrt{3}}{2}$ (t) $\sqrt{6}$ (u) $\frac{\sqrt{3}}{6}$ (v) $\frac{3}{4}$ (w) 1 (x) $\frac{1}{2\sqrt{2}}$

(2) (a) $45°$ (b) $45°$ (c) $60°$ (d) $30°$ (e) $45°$ (f) $30°$ (g) $60°$ (h) $45°$ (i) $30°$ (j) $60°$ (k) $50°$ (l) $50°$ (m) $50°$
(n) $50°$ (o) $30°$ (p) $60°$ (q) $45°$ (r) $60°$ (s) $45°$ (t) $30°$

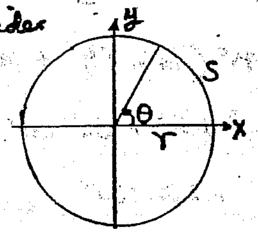
(3) (a) $-\frac{1}{\sqrt{2}}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$ (d) $\sqrt{2}$ (e) $\frac{\sqrt{3}}{2}$ (f) $\frac{1}{\sqrt{3}}$ (g) $-\frac{1}{\sqrt{2}}$ (h) $-\frac{\sqrt{3}}{2}$ (i) 0 (j) $\frac{1}{2}$ (k) 5 (l) 0
(m) 9 (n) -2

RADIAN MEASURE IN TRIGONOMETRY

THE DEFINITION OF RADIAN MEASURE

a **RADIAN** is the unit of circular measure of an angle θ subtended at the center of a circle with **radius** r by an **arc** S equal in length to the radius.

consider



Then, if $S = r$, $\theta = 1$ radian.

Generally $\boxed{\theta = \frac{S}{r} \text{ RADIANS}}$

NOTE: 1 radian = 57.30°

THE CONVERSION RELATIONSHIPS

Consider that $360^\circ = \frac{2\pi r}{r} = 2\pi$ radians

thus $\boxed{180^\circ = \pi \text{ radians}}$ then

$\boxed{1^\circ = \frac{\pi}{180} \text{ radians}}$ and $\boxed{1 \text{ radian} = \frac{180^\circ}{\pi}}$

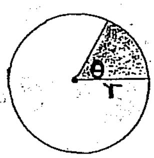
EQUIVALENT ANGLES OF INTEREST

$\boxed{30^\circ = \frac{\pi}{6} \text{ radians}}$ $\boxed{45^\circ = \frac{\pi}{4} \text{ radians}}$ $\boxed{60^\circ = \frac{\pi}{3} \text{ radians}}$

$0^\circ = 0$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$, $270^\circ = \frac{3\pi}{2}$, $360^\circ = 2\pi$

THE AREA OF A SECTOR

Consider the **AREA**, A, of the sector of the circle, as shaded below:



then $\boxed{A = \frac{1}{2} r^2 \theta \text{, } \theta \text{ in radians}}$

RADIAN MEASURE IN TRIGONOMETRY - EXAMPLES

① Convert 315° to radians.

consider $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$ radians

② Convert $\frac{8\pi}{3}$ to degrees.

consider $\frac{8\pi}{3} \times \frac{180}{\pi} = 480°$

③ Find the exact value of: ⓐ $\sin \frac{7\pi}{4}$ ⓑ $\cos \frac{8\pi}{3}$

ⓐ $\sin \frac{7\pi}{4} = \sin 315°$, see conversion above

$= \pm \sin 45°$, reference angle is 45° ($\frac{\pi}{4}$ radians)

$= -\frac{1}{\sqrt{2}}$, special angle result and CAST

ⓑ $\cot \frac{8\pi}{3} = \cot 480°$, see conversion above

$= \cot 120°$, co-terminal angles

$= \pm \cot 60°$, reference angle is 60° ($\frac{\pi}{3}$ radians)

$= -\frac{1}{\sqrt{3}}$, special angle result and CAST

④ Find the central angle, in radians and degrees, subtended by a 45 cm arc of a circle with radius 18 cm.

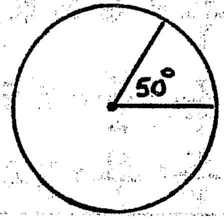
consider that $\theta = \frac{s}{r}$ radians, by definition

then $\theta = \frac{45\text{ cm}}{18\text{ cm}}$ radians

$\therefore \theta = 2.5$ radians

and $\theta = \left(2.5 \times \frac{180}{\pi}\right)° = 143.24°$

⑤ Find the area of the sector below if the radius of the circle is 15 cm.

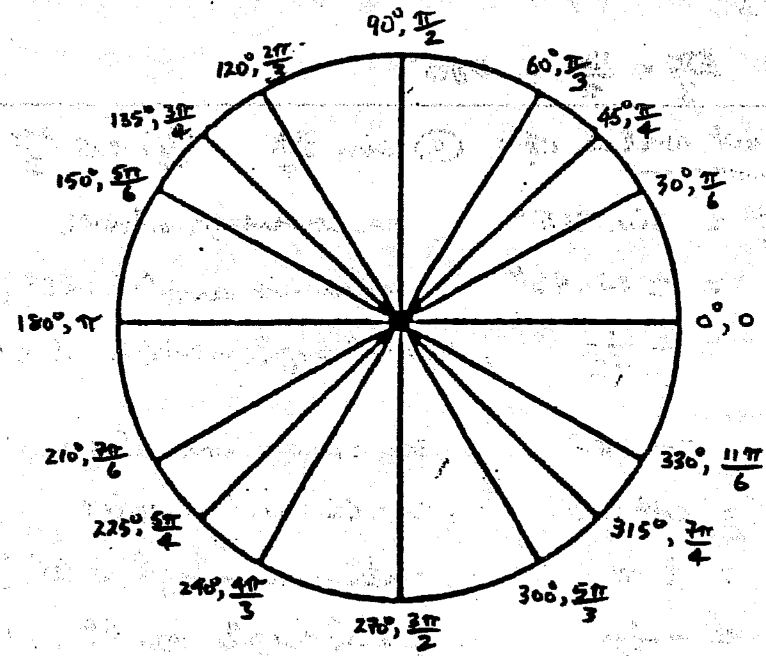


consider $A = \frac{1}{2} r^2 \theta$, θ in radians

then $A = \frac{1}{2}(15)^2 \left[50 \times \frac{\pi}{180}\right]$

$\therefore A = 98.2 \text{ cm}^2$

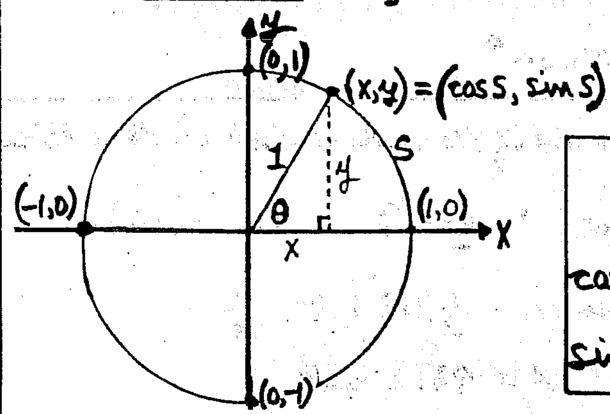
For convenience, we include the following summary of the angles (in degrees and radians) for which exact values of the trigonometric functions are available.



NOTE: In advanced treatments (like calculus) we consider the trigonometric functions as CIRCULAR FUNCTIONS defined on the unit circle as follows:

THE UNIT CIRCLE AND CIRCULAR FUNCTIONS

The unit circle $x^2+y^2=1$



then

$$\theta = \frac{s}{r} = \frac{s}{1} = s \in \mathbb{R}$$

$$\cos\theta = \cos s = \frac{x}{r} = \frac{x}{1} = x \in \mathbb{R}$$

$$\sin\theta = \sin s = \frac{y}{r} = \frac{y}{1} = y \in \mathbb{R}$$

Hence, the domains and ranges of the trigonometric functions consist of real numbers.

RADIAN MEASURE IN TRIGONOMETRY – EXERCISES

(1) What is the radian measure of a central angle where:

(a) $s = r$ (b) $s = 2r$ (c) $2s = r$ (d) $s = 10,\ r = 2.5$?

(2) Convert to radians:

(a) $45°$ (b) $270°$ (c) $40°$ (d) $75°$ (e) $85°$ (f) $105°$ (g) $135°$ (h) $160°$ (i) $165°$ (j) $210°$
(k) $225°$ (l) $250°$ (m) $300°$ (n) $315°$ (o) $330°$ (p) $390°$ (q) $580°$ (r) $1260°$ (s) $-720°$
(t) $-375°$ (u) $-450°$ (v) $-120°$ (w) $930°$ (x) $213.50°$ (y) $111.75°$ (z) $5000°$

(3) Convert to degrees:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{5}$ (d) $\frac{5\pi}{18}$ (e) $\frac{7\pi}{15}$ (f) $\frac{7\pi}{6}$ (g) $\frac{6\pi}{5}$ (h) $\frac{5\pi}{3}$ (i) $\frac{10\pi}{3}$ (j) $\frac{7\pi}{3}$ (k) $\frac{27\pi}{5}$
(l) $-\frac{5\pi}{12}$ (m) $-\frac{4\pi}{5}$ (n) 15π (o) 50π (p) 5 (q) 1.38

(4) Find the reference angle (in radians) for each angle:

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$ (e) $\frac{5\pi}{4}$ (f) $\frac{4\pi}{3}$ (g) $\frac{7\pi}{6}$ (h) $\frac{7\pi}{4}$ (i) $\frac{11\pi}{6}$ (j) $\frac{5\pi}{3}$
(k) $\frac{\pi}{5}$ (l) $\frac{8\pi}{5}$ (m) $\frac{9\pi}{4}$ (n) $\frac{10\pi}{3}$ (o) $-\frac{5\pi}{4}$ (p) $-\frac{11\pi}{6}$ (q) $-\frac{10\pi}{3}$

(5) Find the exact value of:

(a) $\sin\frac{5\pi}{4}$ (b) $\cos\frac{5\pi}{6}$ (c) $\tan\frac{5\pi}{3}$ (d) $\csc\frac{\pi}{4}$ (e) $\sec\frac{5\pi}{3}$ (f) $\cot\frac{4\pi}{3}$ (g) $\tan\left(-\frac{11\pi}{6}\right)$
(h) $\sec\left(-\frac{21\pi}{2}\right)$ (i) $\sin\left(-\frac{31\pi}{6}\right)$ (j) $\csc^2\left(\frac{17\pi}{4}\right)$ (k) $\tan\frac{9\pi}{4} + \sin\frac{4\pi}{3}$ (l) $\sin\frac{7\pi}{6} + \cos\frac{5\pi}{6}$
(m) $\cos\frac{3\pi}{4} + \csc\frac{4\pi}{3} + \tan\frac{5\pi}{6} + \sin 6\pi$ (n) $\cos\frac{11\pi}{4} + \tan\frac{5\pi}{3}$ (o) $\sin\frac{19\pi}{3} + \tan\frac{21\pi}{4}$
(p) $\tan\pi\cos\frac{3\pi}{2} + \sec 2\pi - \csc\frac{3\pi}{2}$ (q) $\sin\frac{\pi}{3}\cot\frac{\pi}{6}\tan\frac{\pi}{4}$ (r) $\tan^2\left(\frac{\pi}{3}\right)\sec\frac{\pi}{6}\sin\frac{\pi}{4}$
(s) $\sin\frac{\pi}{2}\cos\pi + \tan\frac{\pi}{3}\cot\frac{\pi}{3}$ (t) $\tan^3\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{6}\right)$ (u) $\tan^2\left(\frac{2\pi}{3}\right) + 4\cos^2\left(\frac{7\pi}{4}\right) + 3\sec^2\left(\frac{7\pi}{6}\right)$

(6) Find the central angle (in both radians and degrees) subtended by a

(a) 10 cm arc of a circle with radius 5 cm.
(b) 45 in. arc of a circle with radius 10 in.
(c) 3 cm arc of a circle with radius 4 cm.

(7) Find the radius of a circle where a

(a) 5 cm arc subtends a central angle of $\frac{\pi}{6}$ radians.
(b) 10 cm arc subtends a central angle of $85°$.
(c) 6 in. arc subtends a central angle of $200.45°$.

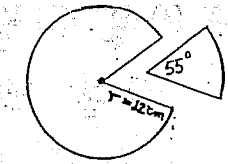
(8) Find the length of the arc of a circle subtended by a central angle of
   (a) 50°, with a 5 cm radius.
   (b) 1.75 radians, with a 3 m radius.
   (c) 140°, with a 6 in. radius.

(9) Find the area of the sector of the circle with central angle θ and radius r:
   (a) $\theta = \frac{\pi}{3}$ radians, r = 9 cm (b) $\frac{3\pi}{10}$ radians, 52 ft. (c) 15°, 8 m
   (d) 125°, 12 in. (e) 81°, r = 12.7 cm.

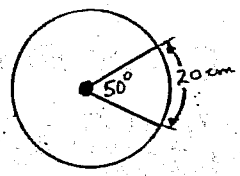
(10) The area of the sector of a circle is 605 cm^2. If the central angle of the sector is 50°, then find the radius of the circle.

(11) Find the central angle (in both radians and degrees) of the sector of a circle of radius 15 cm, if the area of the sector is 65 cm^2.

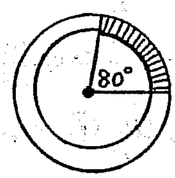
(12) Find the area of the circle that remains if the sector is removed, as shown to the right.



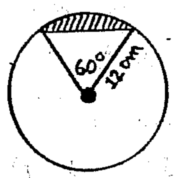
(13) Find the area of the circle:



(14) Find the shaded area if the radii are 15 cm and 13 cm respectively:



*(15) Find the area of the shaded portion in the picture below.



RADIAN MEASURE IN TRIGONOMETRY - EXERCISE - ANSWERS

(1) (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 4

(2) (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{2\pi}{9}$ (d) $\frac{5\pi}{12}$ (e) $\frac{17\pi}{36}$ (f) $\frac{7\pi}{12}$ (g) $\frac{3\pi}{4}$ (h) $\frac{8\pi}{9}$ (i) $\frac{11\pi}{12}$ (j) $\frac{7\pi}{6}$ (k) $\frac{5\pi}{4}$
(l) $\frac{25\pi}{18}$ (m) $\frac{5\pi}{3}$ (n) $\frac{7\pi}{4}$ (o) $\frac{11\pi}{6}$ (p) $\frac{13\pi}{6}$ (q) $\frac{29\pi}{9}$ (r) 7π (s) -4π (t) $-\frac{25\pi}{12}$ (u) $-\frac{5\pi}{2}$
(v) $-\frac{2\pi}{3}$ (w) $\frac{31\pi}{6}$ (x) 3.73 (y) 1.95 (z) 87.3.

(3) (a) 60° (b) 30° (c) 36° (d) 50° (e) 84° (f) 210° (g) 216° (h) 300° (i) 600° (j) 420° (k) 972° (l) −75°
(m) −144° (n) 2700° (o) 9000° (p) 286.48° (q) 79.07°

(4) (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ (e) $\frac{\pi}{4}$ (f) $\frac{\pi}{3}$ (g) $\frac{\pi}{6}$ (h) $\frac{\pi}{4}$ (i) $\frac{\pi}{6}$ (j) $\frac{\pi}{3}$ (k) $\frac{\pi}{5}$ (l) $\frac{3\pi}{5}$
(m) $\frac{\pi}{4}$ (n) $\frac{\pi}{3}$ (o) $\frac{\pi}{4}$ (p) $\frac{\pi}{6}$ (q) $\frac{\pi}{3}$

(5) (a) $-\frac{1}{\sqrt{2}}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$ (d) $\sqrt{2}$ (e) $\frac{\sqrt{3}}{2}$ (f) $\frac{1}{\sqrt{3}}$ (g) $\frac{1}{\sqrt{3}}$ (h) undefined (i) $\frac{1}{2}$ (j) 2
(k) $\frac{2-\sqrt{3}}{2}$ (l) 0 (m) $\frac{-\sqrt{2}-2\sqrt{3}}{2}$ (n) $-\frac{7\sqrt{2}}{2}$ (o) $\frac{2+\sqrt{3}}{2}$ (p) 2 (q) $\frac{3}{2}$ (r) $\sqrt{6}$ (s) 0
(t) $\frac{7}{4}$ (u) 9

(6) (a) 2 rads., 114.59° (b) 4.5 rads, 257.83° (c) 0.75 rads, 42.97°

(7) (a) $\frac{30}{\pi}$ cm (b) 6.74 cm (c) 1.72 in.

(8) (a) 4.36 cm (b) 5.25 m (c) 4.19 in.

(9) (a) 42 cm^2 (b) 1274.23 $ft.^2$ (c) 8.3776 m^2 (d) 157 in^2 (e) 114 cm^2

(10) 37.2 cm

(11) 0.58 rads, and 30.60°

(12) 383.27 cm^2

(13) $\frac{72}{\pi} \simeq 22.92$ cm^2

(14) $\frac{112\pi}{9} \simeq 39.1$ cm^2

(15) $24\pi - 36\sqrt{3} \simeq 13.04$ cm^2