**LAST NAME:** SOLUTIONS

**FIRST NAME:** ______________

**STUDENT NUMBER:** ______________

# TEST 2 (A)

DAWSON COLLEGE

201-NYC-05 - Linear Algebra

Instructor: E. Richer

Date: July 3rd 2008

**Question 1.** *(10 marks)*

Solve the matrix equation. (Note: $X$ is a matrix, not a scalar)

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 9 & 7 \\ -3 & -1 \\ 14 & 10 \end{bmatrix}$$

X is A 2X2 MATRIX

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ -a & -b \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ -3 & -1 \\ 14 & 10 \end{bmatrix}$$

By observation $a = 3$
$b = 1$

$$\begin{aligned} a+3c &= 9 \\ 3+3c &= 9 \\ 3c &= 6 \\ c &= 2 \end{aligned} \qquad \begin{aligned} b+3d &= 7 \\ 1+3d &= 7 \\ 3d &= 6 \\ d &= 2 \end{aligned}$$

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

CHECK: $\begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ -3 & -1 \\ 14 & 10 \end{bmatrix}$

**Question 2.** *(10 marks)*
The following system of linear equations has a unique solution. Find the solution using the inverse of the coefficient matrix. (Note: No other method will earn you any points)

$x+y+z=5$
$x+y-4z=10$
$-4x+y+z=0$

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

Find $A^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 + 4R_1 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} 5R_1 \to R_1 \\ R_2 + R_3 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 5 & 5 & 5 & 5 & 0 & 0 \\ 0 & 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$$R_1 + R_3 \to R_1 \left[\begin{array}{ccc|ccc} 5 & 5 & 0 & 4 & 1 & 0 \\ 0 & 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right] \quad R_1 - R_2 \to R_1$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & -1 \\ 0 & 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix}$$

$$x = A^{-1}b$$

$$= \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

THE SOLUTION IS $(x, y, z) = (1, 5, -1)$

**Question 3.** *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Find elementary matrices $E_1, E_2$ and $E_3$ that satisfy the following.

(i) $E_1 A = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(ii) $E_2 A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ 3 & 3 & 3 \end{bmatrix}$

(ii) $E_3 A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(i) THE RESULTING MATRIX IS $R_1 \longleftrightarrow R_3$ of $A$

So $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(ii) $2R_2 \rightarrow R_2$ of $A$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $R_3 - R_2 \rightarrow R_3$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

**Question 4.** *(10 marks)*

Let $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 2 & 4 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

$R_2 - 2R_1 \to R_2$
$R_3 + R_1 \to R_3$
$R_4 - 3R_1 \to R_4$
$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & -3 & 10 & -2 \end{bmatrix}$

$R_3 + R_4 \to R_4$ $\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 2 & 4 & -1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

$\frac{1}{9}R_4 \to R_4$
$\frac{1}{2}R_2 \to R_2$
$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 - 3R_2 \to R_3$ $\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & -7 & \frac{7}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 + 7R_4 \to R_3$ $\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 \leftrightarrow R_4$ $\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{7}{2} \end{bmatrix} = B$

$\det B = 7/2$

$\det B = \frac{1}{9} \cdot \frac{1}{2} (-1) \det A$

$\det A = \frac{7}{2} (9)(2)(-1) = -63$

$\boxed{\det A = -63}$

**Question 5.** *(10 marks)*
Let $A$ and $B$ be 4x4 matrices with $\det A = 2$ and $\det B = -2$.
Compute the following.
(i) $\det(2AB^T)$
(ii) $\det((3A)^{-1}(2B)^T)$

(i) $\det(2AB^T) = 2^4 \det(AB^T)$

$= 2^4 \det A \det B^T$

$= 2^4 \det A \det B$

$= 2^4 (2)(-2) = -2^6 = \boxed{-64}$

(ii) $\det((3A)^{-1}(2B)^T)$

$= \det(\frac{1}{3}A^{-1} 2 B^T)$

$= \det(\frac{2}{3}A^{-1}B^T)$

$= \left(\frac{2}{3}\right)^4 \det A^{-1} \det B^T$

$= \left(\frac{2}{3}\right)^4 \frac{1}{\det A} \det B$

$= \frac{2^4}{3^4}\left(\frac{1}{2}\right)(-2)$

$= \boxed{-2^4/3^4}$

**Question 6.** *(10 marks)*
Given $n$ x $n$ matrices $A$ and $B$. State whether or not the following statements are TRUE or FALSE. If the statement is false give an example otherwise simply state that the statement is true.
(a) $\det(A+B) = \det A + \det B$

$\boxed{\text{FALSE}}$ $A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ $A+B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$\det A = -1$
$\det B = -3$
$\det(A+B) = -5$

$\det(A+B) = -5$
$\neq \det A + \det B = -4$

(b) $\det(2A) = 2\det A$

$\boxed{\text{FALSE}}$ Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\det A = 1$

$2A = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$ $\det(2A) = 4$

$\det(2A) \neq 2\det A = 2$

(c) $\det(AB) = \det(BA)$

$\boxed{\text{TRUE}}$

**Question 7.** *(10 marks)*
Prove that if $A$ is an nxn matrix, then $\det(adjA) = (\det A)^{n-1}$

We KNOW

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

TAKING determinants OF BOTH sides

$$\det A^{-1} = \det\left(\frac{1}{\det A} \operatorname{adj} A\right)$$

$$\frac{1}{\det A} = \left(\frac{1}{\det A}\right)^n \det \operatorname{adj} A$$

(pull out the constant $\frac{1}{\det A}$ from within det)

$$\frac{1}{\det A} (\det A)^n = \det \operatorname{adj} A$$

$$\det \operatorname{adj} A = (\det A)^{n-1}$$

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**Question 8.** *(10 marks)*

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where $\det A = 5$. Find $\det B$ where $B = \begin{bmatrix} a-g & b-h & c-i \\ d+2a & e+2b & f+2i \\ -3g & -3h & -3i \end{bmatrix}$.

$$B = \begin{bmatrix} a-g & b-h & c-i \\ d+2a & e+2b & f+2i \\ -3g & -3h & -3i \end{bmatrix} \quad -\frac{1}{3}R_3 \to R_3 \quad \begin{bmatrix} a-g & b-h & c-i \\ d+2a & e+2b & f+2i \\ g & h & i \end{bmatrix}$$

$$R_1 + R_3 \to R_1 \quad \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2i \\ g & h & i \end{bmatrix}$$

$$R_2 - 2R_1 \to R_2 \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$$

$$\det A = -\frac{1}{3} \det B$$

$$\det B = -3 \det A$$

$$= -3(5) = \boxed{-15}$$