NAME: SOLUTIONS - BLUE

TEST #2

201-NYB-05 C16
(Calculus II Science)

Friday, March 28
Time: 11:30 - 12:45

Justify all your work. No documentation is allowed. Non-graphic calculators are permitted. Read every question carefully. Exact values are expected throughout (except in the "work" problem). Write your solutions properly. Test is marked on 100 marks.

1. (**15 marks**) Find the length of the curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$ for $2 \leq x \leq 4$.

$$f(x) = \frac{x^2}{2} - \frac{\ln x}{4}$$

$$f'(x) = x - \frac{1}{4x}$$

$$1 + [f'(x)]^2 = 1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2}$$

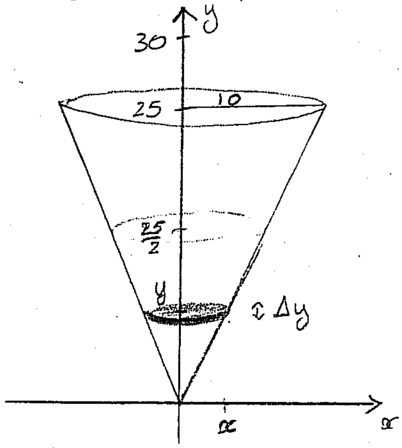
$$= \left(x + \frac{1}{4x}\right)^2$$

$$s = \int_2^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\ln|x|\right]_2^4$$

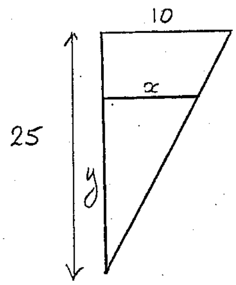
$$= 8 + \frac{1}{4}\ln 4 - 2 - \frac{1}{4}\ln 2 = 6 + \frac{2}{4}\ln 2 - \frac{1}{4}\ln 2$$

$$= 6 + \frac{1}{4}\ln 2.$$

2. (**15 marks**) A conical tank which is 25 feet deep and has a circular top 20 feet in diameter, is half-full. Find the work done in emptying the tank by pumping the water to a height 5 feet above the top of the tank. (The water weighs 62.4 pounds per cubic foot.)



$$\Delta F = (62.4)\underbrace{(\pi x^2 \Delta y)}_{\text{volume of the layer}}$$



$$\frac{10}{25} = \frac{x}{y} \quad \text{so} \quad x = \frac{2}{5}y$$

$$\Delta F = 62.4\,\pi \left(\frac{2}{5}y\right)^2 \Delta y$$

Distance : $(30-y)$ feet

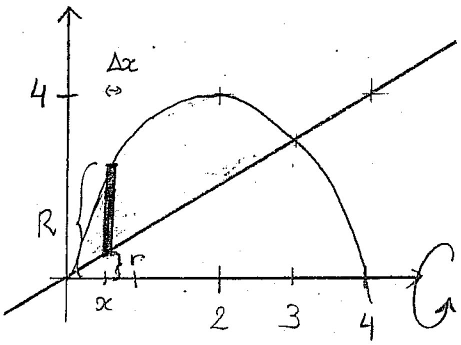
$$\Delta W = (30-y)\, 62.4\,\pi \frac{4}{25} y^2 \Delta y$$

$$W = \int_0^{25/2} 62.4\pi \frac{4}{25}(30-y)y^2\,dy = 9.984\,\pi \int_0^{25/2} (30y^2 - y^3)\,dy$$

$$= 9.984\,\pi \left[10y^3 - \frac{y^4}{4}\right]_0^{25/2} = 9.984\,\pi\left(10(12.5)^3 - \frac{(12.5)^4}{4}\right)$$

$\approx 421\,170$ foot-pounds.

3. (**15 marks**) Sketch the region bounded by $y = 4x - x^2$ and $x - y = 0$. Find the volume of the solid obtained by rotating this region about the x-axis.



Points of intersection:

$$4x - x^2 = x$$
$$3x = x^2$$

so $x = 0$ and $x = 3$.

$r(x) = x \qquad R(x) = 4x - x^2$

$$V = \pi \int_0^3 \left[(4x - x^2)^2 - x^2 \right] dx = \pi \int_0^3 \left[16x^2 - 8x^3 + x^4 - x^2 \right] dx$$

$$= \pi \int_0^3 (15x^2 - 8x^3 + x^4)\, dx = \pi \left[5x^3 - 2x^4 + \frac{x^5}{5} \right]_0^3$$

$$= \pi \left(135 - 162 + \frac{243}{5} \right) = \frac{108}{5}\pi$$

4. **(10 marks)** Evaluate the following integral $I = \int_1^3 \frac{3x}{(x-4)^2}\, dx$

Let $u = x - 4$ $\qquad x = u + 4 \qquad$ when $x = 1$, $u = -3$

$du = dx \qquad\qquad\qquad\qquad\qquad x = 3$, $u = -1$

$$I = \int_{-3}^{-1} \frac{3(u+4)}{u^2}\, du = 3\int_{-3}^{-1} \left(\frac{1}{u} + \frac{4}{u^2}\right) du = 3\left[\ln|u| - \frac{4}{u}\right]_{-3}^{-1}$$

$$= 3\left(\ln 1 - \frac{4}{-1}\right) - 3\left(\ln 3 - \frac{4}{-3}\right) = 0 + 12 - 3\ln 3 - 4$$

$$= 8 - 3\ln 3$$

5. (**10 marks**) Evaluate the following integral $I = \int_1^4 \sqrt{t} \ln t \, dt$

Let $u = \ln t \qquad du = \frac{1}{t}\, dt$

$v = \frac{2}{3} t^{3/2} \qquad dv = t^{1/2}\, dt$

$$I = \left[(\ln t) \frac{2}{3} t^{3/2} \right]_1^4 - \frac{2}{3} \int_1^4 t^{1/2}\, dt$$

$$= \ln 4 \left(\frac{16}{3} \right) - 0 - \frac{2}{3} \left[\frac{2}{3} t^{3/2} \right]_1^4$$

$$= \frac{32}{3} \ln 2 - \frac{4}{9}(8-1)$$

$$= \frac{32}{3} \ln 2 - \frac{28}{9}$$

6. **(10 marks)** Evaluate the following integral $I = \int_{\pi/3}^{\pi/2} \sin^3 x \sqrt{\cos x}\, dx$

$$I = \int_{\pi/3}^{\pi/2} \sin^2 x \sqrt{\cos x}\, \sin x\, dx = \int_{\pi/3}^{\pi/2} (1-\cos^2 x)\sqrt{\cos x}\, \sin x\, dx$$

let $u = \cos x$ $\qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$du = -\sin x\, dx$

$(-1)\, du = \sin x\, dx$ $\qquad \cos\left(\frac{\pi}{2}\right) = 0$

$$I = \int_{1/2}^{0} (1-u^2)\sqrt{u}\,(-1)\, du = -\int_{1/2}^{0} \left(u^{1/2} - u^{5/2}\right) du$$

$$= -\left[\frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}\right]_{1/2}^{0} = +\left(\frac{2}{3}\left(\frac{1}{2}\right)^{3/2} - \frac{2}{7}\left(\frac{1}{2}\right)^{7/2}\right)$$

$$= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right) - \frac{2}{7}\left(\frac{1}{8}\cdot\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{56} = \frac{25\sqrt{2}}{168}$$

7. (**10 marks**) Evaluate the following integral $I = \int_1^2 t^4 (\ln t)^2 \, dt$

Let $u = (\ln t)^2 \qquad du = 2(\ln t) \cdot \frac{1}{t}\, dt$

$v = \frac{t^5}{5} \qquad dv = t^4\, dt$

$$I = \left[\frac{t^5}{5}(\ln t)^2\right]_1^2 - \frac{2}{5}\int_1^2 t^4 \ln t \, dt$$

let $u = \ln t \qquad du = \frac{1}{t}\, dt$

$v = \frac{t^5}{5} \qquad dv = t^4\, dt$

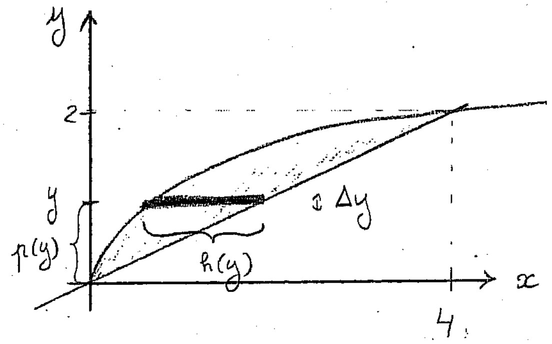
$$I = \left[\frac{32}{5}(\ln 2)^2 - 0\right] - \frac{2}{5}\left(\left[\frac{t^5}{5}\ln t\right]_1^2 - \frac{1}{5}\int_1^2 t^4\, dt\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left(\frac{32}{5}\ln 2 - 0 - \frac{1}{5}\left[\frac{t^5}{5}\right]_1^2\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{64}{125} - \frac{2}{125}$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}.$$

8. **(15 marks)** Sketch the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$. Using the cylindrical shells method, find the volume of the solid obtained by rotating this region about the x-axis.



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Points of intersection

$$\sqrt{x} = \frac{x}{2}$$

$$x = \frac{x^2}{4}$$

so $x = 0$ and $x = 4$

that is $y = 0$ and $y = 2$

$$y = \sqrt{x} \Rightarrow x = y^2$$

$$y = \frac{x}{2} \Rightarrow x = 2y$$

$p(y) = y$ and $h(y) = 2y - y^2$

$$V = 2\pi \int_0^2 y(2y - y^2)\,dy = 2\pi \int_0^2 (2y^2 - y^3)\,dy$$

$$= 2\pi \left[2\frac{y^3}{3} - \frac{y^4}{4} \right]_0^2 = 2\pi \left(\frac{16}{3} - 4 \right) = \frac{8\pi}{3}.$$