NAME: SOLUTIONS - IVORY

TEST #2

201-NYB-05 C16
(Calculus II Science)

Friday, March 28
Time: 11:30 - 12:45

Justify all your work. No documentation is allowed. Non-graphic calculators are permitted. Read every question carefully. Exact values are expected throughout (except in the "work" problem). Write your solutions properly. Test is marked on 100 marks.

1. (**10 marks**) Evaluate the following integral $I = \int_0^{\pi/2} \sec^4\left(\frac{t}{2}\right)\, dt$

$$I = \int_0^{\pi/2} \sec^2\left(\frac{t}{2}\right) \sec^2\left(\frac{t}{2}\right) dt = \int_0^{\pi/2} \left(1 + \tan^2\left(\frac{t}{2}\right)\right) \sec^2\left(\frac{t}{2}\right) dt$$

let $u = \tan\left(\frac{t}{2}\right)$

$du = \frac{1}{2} \sec^2\left(\frac{t}{2}\right) dt$

$2\,du = \sec^2\left(\frac{t}{2}\right) dt$

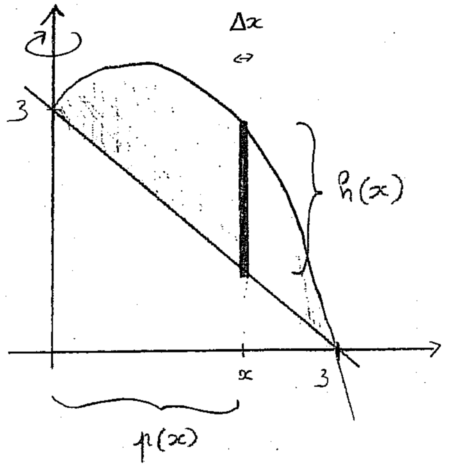
$\tan\left(\frac{1}{2} \cdot \frac{\pi}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$\tan\left(\frac{1}{2} \cdot 0\right) = 0$

$$I = 2\int_0^1 (1 + u^2)\, du = 2\left[u + \frac{u^3}{3}\right]_0^1 = 2\left(1 + \frac{1}{3}\right)$$

$$= \frac{8}{3}.$$

2. **(15 marks)** Sketch the region bounded by $y = 3 + 2x - x^2$ and $x + y = 3$. Find the volume of the solid obtained by rotating this region about the $y$-axis.




Points of intersection

$$3 + 2x - x^2 = 3 - x$$

$$x^2 - 3x = 0$$

so $x = 0$ and $x = 3$

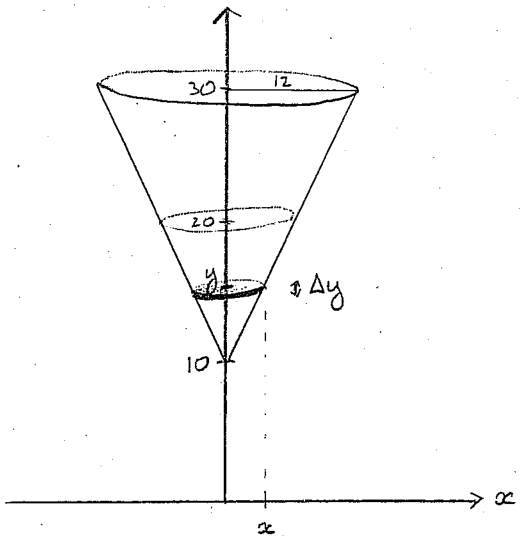
$r(x) = x$

$h(x) = (3 + 2x - x^2) - (3 - x) = 3x - x^2$

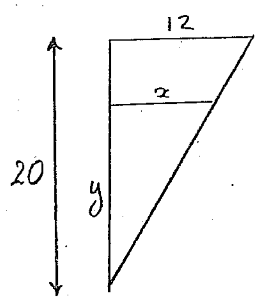
$$V = 2\pi \int_0^3 x(3x - x^2)\,dx = 2\pi \int_0^3 (3x^2 - x^3)\,dx$$

$$= 2\pi \left[ x^3 - \frac{x^4}{4} \right]_0^3 = 2\pi \left( 27 - \frac{81}{4} \right) = \frac{27}{2}\pi.$$

3. **(15 marks)** A conical tank is 20 feet deep and has a circular top 12 feet in diameter. Find the work done in filling half the tank from a source 10 feet below the bottom of the tank. (The water weighs 62.4 pounds per cubic foot.)




$$\Delta F = (62.4)\underbrace{(\pi x^2 \Delta y)}_{\text{volume of the layer}}$$




$$\frac{12}{20} = \frac{x}{y} \quad \text{so} \quad x = \frac{3}{5}y$$

$$\Delta F = 62.4\,\pi\left(\frac{3}{5}y\right)^2 \Delta y$$

Distance : $y$ feet

$$\Delta W = y \cdot 62.4\,\pi\,\frac{9}{25}y^2\,\Delta y$$

$$W = \int_{10}^{20} 62.4\,\pi\,\frac{9}{25}\,y^3\,dy = 22.464\,\pi\left[\frac{y^4}{4}\right]_{10}^{20}$$

$$= 22.464\,\pi\,(40\,000 - 2\,500)$$

$$\approx 2\,646\,478 \text{ foot-pounds.}$$

4. **(15 marks)** Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$ for $2 \le x \le 4$.

$$f(x) = \frac{x^2}{2} - \frac{\ln x}{4}$$

$$f'(x) = x - \frac{1}{4x}$$

$$1 + [f'(x)]^2 = 1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}$$

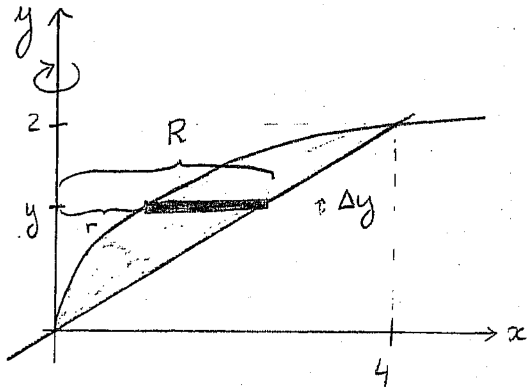
$$= x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$s = \int_2^4 \sqrt{1 + [f'(x)]^2}\, dx = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\ln|x|\right]_2^4$$

$$= 8 + \frac{1}{4}\ln 4 - 2 - \frac{1}{4}\ln 2 = 6 + \frac{2}{4}\ln 2 - \frac{1}{4}\ln 2$$

$$= 6 + \frac{1}{4}\ln 2.$$

5. **(15 marks)** Sketch the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$. Using the disk/washer method, find the volume of the solid obtained by rotating this region about the $y$-axis.




Points of intersection

$$\sqrt{x} = \frac{x}{2}$$

$$x = \frac{x^2}{4}.$$

so $x = 0$ and $x = 4$

that is $y = 0$ and $y = 2$.

$y = \sqrt{x} \Rightarrow x = y^2$

$y = \frac{x}{2} \Rightarrow x = 2y$

$r(y) = y^2 \qquad R(y) = 2y$

$$V = \pi \int_0^2 \left[ (2y)^2 - (y^2) \right] dy = \pi \int_0^2 3y^2 \, dy$$

$$= \pi \left[ y^3 \right]_0^2 = 8\pi$$

6. (**10 marks**) Evaluate the following integral $I = \int_0^1 x^2 e^{-x}\, dx$

Let $u = x^2 \qquad du = 2x\,dx$

$v = -e^{-x} \qquad dv = e^{-x}\,dx$

$$I = \left[-x^2 e^{-x}\right]_0^1 - \int_0^1 -2x e^{-x}\,dx$$

$$= -e^{-1} - 0 + 2\int_0^1 x e^{-x}\,dx$$

Let $u = x \qquad du = dx$

$v = -e^{-x} \qquad dv = e^{-x}\,dx$

$$I = -\frac{1}{e} + 2\left(\left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x}\,dx\right)$$

$$= \frac{-1}{e} + 2\left(-e^{-1} - 0 + \int_0^1 e^{-x}\,dx\right)$$

$$= \frac{-1}{e} + 2\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right)$$

$$= \frac{-1}{e} - \frac{2}{e} - 2\left(e^{-1} - e^{-0}\right)$$

$$= \frac{-3}{e} - \frac{2}{e} + 2$$

$$= 2 - \frac{5}{e}.$$

7. **(10 marks)** Evaluate the following integral $I = \int_1^2 x^4 (\ln x)^2 \, dx$

Let $u = (\ln x)^2$ $\quad du = 2(\ln x)\frac{1}{x}\,dx$

$v = \frac{x^5}{5}$ $\quad dv = x^4\,dx$

$$I = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - \frac{2}{5}\int_1^2 x^4 \ln x \, dx$$

let $u = \ln x$ $\quad du = \frac{1}{x}\,dx$

$v = \frac{x^5}{5}$ $\quad dv = x^4\,dx$

$$I = \left(\frac{32}{5}(\ln 2)^2 - 0\right) - \frac{2}{5}\left(\left[\frac{x^5}{5}\ln x\right]_1^2 - \frac{1}{5}\int_1^2 x^4\,dx\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left(\frac{32}{5}\ln 2 - 0 - \frac{1}{5}\left[\frac{x^5}{5}\right]_1^2\right)$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{64}{125} - \frac{2}{125}$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}.$$

8. (**10 marks**) Evaluate the following integral $I = \int_0^2 \frac{2t}{(t-3)^2}\,dt$

Let $u = t - 3$ $\quad t = u + 3$ $\quad$ when $t = 0$, $u = -3$

$du = dt$ $\quad\quad t = 2$, $u = -1$

$$I = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du = 2\int_{-3}^{-1}\left(\frac{1}{u} + \frac{3}{u^2}\right)du$$

$$= 2\left[\ln|u| - \frac{3}{u}\right]_{-3}^{-1} = 2\left(\ln 1 - \frac{3}{-1}\right) - 2\left(\ln 3 - \frac{3}{-3}\right)$$

$$= 0 + 6 - 2\ln 3 - 2 = 4 - 2\ln 3.$$