Dawson College: **Calculus II**: 201-NYB-05-C16: Winter 2008

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Test 3

This test is graded out of 100 marks. No books, notes, no graphing calculator or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(10 marks)* Integrate the following indefinite integral:

$$\int \frac{x^2-1}{x^3+x}\,dx = \int \frac{x^2-1}{x(x^2+1)}\,dx$$

Lets use partial fractions:

$$\frac{x^2-1}{x^3+x} = \frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2-1 = A(x^2+1) + (Bx+C)x$$

Let $x=0$

$$-1 = A((0)^2+1) + (Bx+C)(0)$$
$$-1 = A$$

Let $x=1$

$$(1)^2-1 = (-1)((1)^2+1) + (B(1)+C)(1)$$
$$0 = -2 + B + C$$
$$① \quad 2-B = C$$

Let $x=-1$

$$(-1)^2-1 = (-1)((-1)^2+1) + (B(-1)+C)(-1)$$
$$0 = -2 + (+B-C)$$
$$② \quad 2+C = B \qquad C = 2-(2+C) \implies C=0$$
$$\implies B=2$$

Using ① and ②

$$\int \frac{x^2-1}{x^3+x}\,dx = \int \frac{-1}{x} + \frac{2x}{x^2+1}\,dx$$

$$= -\ln|x| + \ln|x^2+1| + C$$

Question 2. *(15 marks)* Integrate the following indefinite integral:

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

① $x = 2\sec\theta$ $\Rightarrow \frac{x}{2} = \sec\theta$

② $dx = 2\sec\theta\tan\theta$

$$= \int \frac{\sqrt{(2\sec\theta)^2-4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{4(\sec^2\theta-1)}\;\tan\theta\,d\theta$$

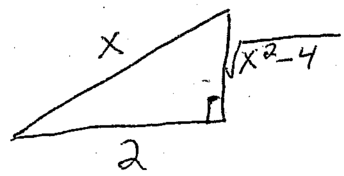
$$= \int \sqrt{\cancel{4}\tan^2\theta}\;\tan\theta\,d\theta$$

$$= 2\int \tan^2\theta\,d\theta$$

$$= 2\int (\sec^2\theta - 1)\,d\theta$$

$$= 2\tan\theta - 2\theta + C$$

$$= \sqrt{x^2-4} - 2\,\mathrm{arcsec}\,\frac{x}{2} + C$$



$$\tan\theta = \frac{\sqrt{x^2-4}}{2}$$

$$\theta = \mathrm{arcsec}\,\frac{x}{2}$$

Question 3. *(15 marks)* Use the Trapezoidal Rule and Simpson's Rule to approximate the value of the following definite integral for $n = 4$. Round your answer to six decimal places and compare the results to the exact value of the definite integral.

$$\int_0^2 x\sqrt{x^2+1}\,dx = \frac{(x^2+1)^{3/2}}{3}\Big|_0^2 = \frac{((2)^2+1)^{3/2}}{3} - \frac{(0^2+1)^{3/2}}{3} = \frac{(\sqrt{5})^3-1}{3}$$

$$\doteq 3.393447$$

$$\Delta x = \frac{b-a}{4} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \quad x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,$$

$$x_3 = \frac{3}{2},\ x_4 = 2$$

Trapezoidal Rule:

$$\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{2}{8}\left[0\sqrt{0^2+1} + 2\left(\tfrac{1}{2}\right)\sqrt{\left(\tfrac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\tfrac{3}{2}\right)\sqrt{\left(\tfrac{3}{2}\right)^2+1}\right.$$

$$\left. + 2\sqrt{2^2+1}\right]$$

$$= \frac{1}{4}\left[\sqrt{5/4} + 2\sqrt{2} + 3\sqrt{13/4} + 2\sqrt{5}\right]$$

$$\doteq 3.456731$$

Simpson's Rule:

$$\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{2}{12}\left[0\sqrt{0^2+1} + 4\left(\tfrac{1}{2}\right)\sqrt{\left(\tfrac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\tfrac{3}{2}\right)\sqrt{\left(\tfrac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right]$$

$$= \frac{1}{6}\left[2\sqrt{5/4} + 2\sqrt{2} + 6\sqrt{13/4} + 2\sqrt{5}\right]$$

$$\doteq 3.392214$$

$\therefore$ Simpson's Rule is best for this problem.

Question 4. *(15 marks)* Evaluate the following limit.

$$\lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$$

Let $y = \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$ has indeterminate form 0^0

apply ln on both sides

$$\ln y = \ln \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$$

$$\ln y = \lim_{x\to 0^+} \ln\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x \quad \text{since ln is continuous}$$

$$\ln y = \lim_{x\to 0^+} x\ln\left[\cos\left(\frac{\pi}{2}-x\right)\right] \quad \text{has indeterminate form } 0\cdot-\infty$$

$$\ln y = \lim_{x\to 0^+} \frac{\ln\left[\cos\left(\frac{\pi}{2}-x\right)\right]}{1/x} \quad \text{has indeterminate form } \frac{-\infty}{\infty}$$

therefore we use L'Hopital's Rule.

$$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{\cos\left(\frac{\pi}{2}-x\right)}\cdot -\sin\left(\frac{\pi}{2}-x\right)(-1)}{-\frac{1}{x^2}}$$

$$\ln y = \lim_{x\to 0^+} \frac{-x^2\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} \quad \text{has indeterminate form } \frac{0}{0}$$

we use L'Hopital's Rule.

$$\ln y = \lim_{x\to 0^+} \frac{-2x\sin\left(\frac{\pi}{2}-x\right)+(-x^2)\cos\left(\frac{\pi}{2}-x\right)(-1)}{\sin\left(\frac{\pi}{2}-x\right)}$$

$$\ln y = 0$$

$$y = 1$$

$$\therefore \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x = 1$$

Question 5. *(15 marks)* Evaluate the following improper integral if it converges.

$$\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx$$

infinite discontinuity at $x=2$.

$$= \int_1^2 \frac{2}{(x-2)^{8/3}}\,dx + \int_2^3 \frac{2}{(x-2)^{8/3}}\,dx$$

$$= \lim_{b\to 2^-} \int_1^b \frac{2}{(x-2)^{8/3}}\,dx + \lim_{a\to 2^+} \int_a^3 \frac{2}{(x-2)^{8/2}}\,dx$$

$$= \lim_{b\to 2^-} \left[\frac{-3\cdot 2}{5(x-2)^{5/3}}\right]_1^b + \lim_{a\to 2^+} \left[\frac{-3\cdot 2}{5(x-2)^{5/3}}\right]_a^3$$

$$= \lim_{b\to 2^-} \left[\frac{-6}{5(b-2)^{5/3}} + \frac{6}{5(1-2)^{5/3}}\right] + \lim_{a\to 2^+} \left[\frac{-6}{5(3-2)^{5/3}} + \frac{6}{(a-2)^{5/3}}\right]$$

both limits diverge hence the integral diverges.

Question 6. *(15 marks)* Determine the convergence or divergence of the sequence with the following n^{th} term. If the sequence converges, find its limit.

$$a_n = \frac{n!\, e^{-n/2}}{(n-1)!}$$

$$\lim_{n\to\infty} \frac{n!\, e^{-n/2}}{(n-1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n-1)!}\, n e^{-n/2}}{\cancel{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{n}{e^{n/2}}$$

has indeterminate form $\frac{\infty}{\infty}$

we use L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{n/2}}{2}}$$

$$= 0$$

$\therefore$ the sequence converges to 0.

Question 7. *(15 marks)* Find the sum of the infinite series.

$$\sum_{n=1}^{\infty}\left[\frac{3}{7^{n+1}}-\frac{4}{9^{n-1}}\right]$$ Lets look at both terms independently.

$$\sum_{n=1}^{\infty}\frac{3}{7^{n+1}} = \sum_{n=1}^{\infty}\frac{3}{7\cdot 7^{n}}$$

$$= \sum_{n=1}^{\infty}\frac{3}{7}\left(\frac{1}{7}\right)^{n}$$

$$= \sum_{n=0}^{\infty}\frac{3}{7}\left(\frac{1}{7}\right)^{n} - a_0$$ geometric series converges since $r=\frac{1}{7}<1$

$$= \frac{3/7}{1-1/7} - \frac{3}{7}\left(\frac{1}{7}\right)^{0}$$

$$= \frac{3/7}{6/7} - \frac{3}{7} = \frac{3}{6}-\frac{3}{7} = \frac{1}{14}$$

$$\sum_{n=1}^{\infty}\frac{4}{9^{n-1}} = \sum_{n=1}^{\infty}\frac{4}{9^{n}9^{-1}}$$

$$= \sum_{n=1}^{\infty}\frac{36}{9^{n}}$$

$$= \sum_{n=0}^{\infty}36\left(\frac{1}{9}\right)^{n} - a_0$$ geometric series converges since $r=\frac{1}{9}<1$

$$= \sum_{n=0}^{\infty}36\left(\frac{1}{9}\right) - 36\left(\frac{1}{9}\right)^{0}$$

$$= \frac{36}{1-1/9} - 36 = \frac{36}{8/9} - 36 = \frac{9}{2}$$

$$\therefore \sum_{n=1}^{\infty}\left[\frac{3}{7^{n+1}}-\frac{4}{9^{n-1}}\right] = \sum_{n=1}^{\infty}\frac{3}{7^{n+1}} - \sum_{n=1}^{\infty}\frac{4}{9^{n-1}}$$

$$= \frac{1}{14}-\frac{9}{2} = -\frac{31}{7}$$