Dawson College: Calculus II: 201-NYB-05-C16: Winter 2008

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Test 3

This test is graded out of 100 marks. No books, notes, no graphing calculator or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(10 marks)* Integrate the following indefinite integral:

$$\int \frac{x^2-2}{x^3+x}\,dx = \int \frac{x^2-2}{x(x^2+1)}\,dx$$

Lets use partial fractions:

$$\frac{x^2-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2-2 = A(x^2+1) + (Bx+C)x$$

Let $x=0$

$$-2 = A((0)^2+1) + (B(0)+C)(0)$$

$$-2 = A$$

Let $x=1$

$$(1)^2-2 = A((1)^2+1) + (B(1)+C)(1)$$

$$-1 = -2(2) + B + C$$

$$3 = B + C$$

(1) $3 - B = C$

Let $x=-1$

$$(-1)^2-2 = A((-1)^2+1) + (B(-1)+C)(-1)$$

$$-1 = -2(2) + B - C$$

(2) $C + 3 = B$

Using (1) & (2)

$$C = 3-(C+3)$$

$$C = 0$$

$$\Rightarrow B = 3$$

$$\int \frac{x^2-2}{x(x^2+1)}\,dx = \int \frac{-2}{x} + \frac{3x}{x^2+1}\,dx$$

$$= -2\ln|x| + \frac{3}{2}\ln|x^2+1| + C$$

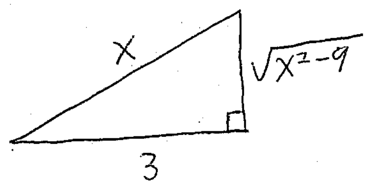
Question 2. *(15 marks)* Integrate the following indefinite integral:

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

① $x = 3\sec\theta$ $\Rightarrow \frac{x}{3} = \sec\theta$

② $dx = 3\sec\theta\tan\theta\, d\theta$

$$= \int \frac{\sqrt{(3\sec\theta)^2-9}}{\cancel{3\sec\theta}}\, \cancel{3\sec\theta}\tan\theta\, d\theta$$



$$= \int \sqrt{9\tan^2\theta}\;\tan\theta\, d\theta$$

$$= \int 3\tan\theta\tan\theta\, d\theta$$

$\tan\theta = \frac{\sqrt{x^2-9}}{3}$

$$= \int 3\tan^2\theta\, d\theta$$

$\theta = \operatorname{arcsec}\frac{x}{3}$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

$$= 3\tan\theta - 3\theta + C$$

$$= \sqrt{x^2-9} - 3\operatorname{arcsec}\frac{x}{3} + C$$

Question 3. *(15 marks)* Use the Trapezoidal Rule and Simpson's Rule to approximate the value of the following definite integral for $n = 4$. Round your answer to six decimal places and compare the results to the exact value of the definite integral.

$$\int_0^2 x\sqrt{x^2+2}\,dx = \left.\frac{(x^2+2)^{3/2}}{3}\right|_0^2 = \frac{(2^2+2)^{3/2}}{3} - \frac{(0^2+2)^{3/2}}{3} = \frac{(6)^{3/2}}{3} - \frac{2^{3/2}}{3}$$

$$\doteq 3.956170$$

$$\Delta x = \frac{b-a}{4} = \frac{2-0}{4} = \frac{1}{2} \qquad x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$$

$$x_3 = \frac{3}{2},\ x_4 = 2$$

Trapezoidal Rule:

$$\int_0^2 x\sqrt{x^2+2}\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{2}{8}\left[0\sqrt{0^2+2} + 2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+2} + 2\sqrt{1^2+2} + 2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+2}\right.$$
$$\left. + 2\sqrt{2^2+2}\right]$$

$$= \frac{1}{4}\left[\sqrt{9/4} + 2\sqrt{3} + 3\sqrt{17/4} + 2\sqrt{6}\right]$$

$$\doteq 4.011935$$

Simpson's Rule:

$$\int_0^2 x\sqrt{x^2+2}\,dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{2-0}{3(4)}\left[0\sqrt{0^2+2} + 4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+2} + 2\sqrt{1^2+2} + 4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+2}\right.$$
$$\left. + 2\sqrt{2^2+2}\right]$$

$$= \frac{1}{6}\left[2\sqrt{9/4} + 2\sqrt{3} + 6\sqrt{17/4} + 2\sqrt{6}\right]$$

$$\doteq 3.955400$$

$\therefore$ Simpson's Rule is best for this problem.

Question 4. *(15 marks)* Evaluate the following limit.

$$\lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$$

Let $y = \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$ has indeterminate form 0^0

apply ln on both sides

$$\ln y = \ln \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x$$

$$\ln y = \lim_{x\to 0^+} \ln\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x \quad \text{since ln is continuous}$$

$$\ln y = \lim_{x\to 0^+} x\ln\left[\cos\left(\frac{\pi}{2}-x\right)\right] \quad \text{has indeterminate form } 0\cdot-\infty$$

$$\ln y = \lim_{x\to 0^+} \frac{\ln\left[\cos\left(\frac{\pi}{2}-x\right)\right]}{1/x} \quad \text{has indeterminate form } \frac{-\infty}{\infty}$$

therefore we use L'Hopital's Rule.

$$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{\cos\left(\frac{\pi}{2}-x\right)}-\sin\left(\frac{\pi}{2}-x\right)(-1)}{-1/x^2}$$

$$\ln y = \lim_{x\to 0^+} \frac{-x^2\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)} \quad \text{has indeterminate form } \frac{0}{0}$$

we use L'Hopital's Rule.

$$\ln y = \lim_{x\to 0^+} \frac{-2x\sin\left(\frac{\pi}{2}-x\right)+(-x^2)\cos\left(\frac{\pi}{2}-x\right)(-1)}{\sin\left(\frac{\pi}{2}-x\right)}$$

$$\ln y = 0$$

$$y = 1$$

$$\therefore \lim_{x\to 0^+}\left[\cos\left(\frac{\pi}{2}-x\right)\right]^x = 1$$

Question 5. *(15 marks)* Evaluate the following improper integral if it converges.

$$\int_1^4 \frac{2}{(x-3)^{10/3}}\, dx$$

Infinite discontinuity at $x=3$.

$$= \int_1^3 \frac{2}{(x-3)^{10/3}}\, dx + \int_3^4 \frac{2}{(x-3)^{10/3}}\, dx$$

$$= \lim_{b \to 3^-} \int_1^b \frac{2}{(x-3)^{10/3}}\, dx + \lim_{a \to 3^+} \int_a^4 \frac{2}{(x-3)^{10/3}}\, dx$$

$$= \lim_{b \to 3^-} \left[\frac{-2 \cdot 3}{7(x-3)^{7/3}} \right]_1^b + \lim_{a \to 3^+} \left[\frac{-2 \cdot 3}{7(x-3)^{7/3}} \right]_a^4$$

$$= \lim_{b \to 3^-} \left[\frac{6}{7(1-3)^{7/3}} - \frac{6}{7(b-3)^{7/3}} \right] + \lim_{a \to 3^+} \left[\frac{6}{7(4-3)^{7/3}} - \frac{6}{7(a-3)^{7/3}} \right]$$

both limits diverges hence the integral diverges.

Question 6. *(15 marks)* Determine the convergence or divergence of the sequence with the following n^{th} term. If the sequence converges, find its limit.

$$a_n = \frac{n!\, e^{-n/3}}{(n-1)!}$$

$$\lim_{n\to\infty} \frac{n!\, e^{-n/3}}{(n-1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n-1)!}\, n\, e^{-n/3}}{\cancel{(n-1)!}}$$

$$= \lim_{n\to\infty} \frac{n}{e^{n/3}}$$ has indeterminate form $\frac{\infty}{\infty}$

we use L'Hopital's Rule

$$= \lim_{n\to\infty} \frac{1}{e^{n/3}/3}$$

$$= 0$$

$\therefore$ the sequence converges to 0.

Question 7. *(15 marks)* Find the sum of the infinite series.

$$\sum_{n=1}^{\infty}\left[\frac{6}{5^{n+1}}-\frac{2}{8^{n-1}}\right]$$

Lets determine the sum of both series independently.

$$\sum_{n=1}^{\infty}\frac{6}{5^{n+1}} = \sum_{n=1}^{\infty}\frac{6}{5\cdot 5^n}$$

$$= \sum_{n=1}^{\infty}\frac{6}{5}\left(\frac{1}{5}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{6}{5}\left(\frac{1}{5}\right)^n - a_0$$

$$= \frac{a}{1-r} - \frac{6}{5}\left(\frac{1}{5}\right)^0$$

$$= \frac{6/5}{1-1/5} - \frac{6}{5} = \frac{6/5}{4/5} = \frac{6}{4}-\frac{6}{5} = \frac{3}{10}$$

$$\sum_{n=1}^{\infty}\frac{2}{8^{n-1}} = \sum_{n=1}^{\infty}\frac{16}{8^n}$$

$$= \sum_{n=1}^{\infty}16\left(\frac{1}{8}\right)^n$$

$$= \sum_{n=a}^{\infty}16\left(\frac{1}{8}\right)^n - a_0$$

$$= \frac{16}{1-1/8} - 16\left(\frac{1}{8}\right)^0$$

$$= \frac{16}{7/8} - 16 = \frac{128}{7} - 16 = \frac{16}{7}$$

$$\sum_{n=1}^{\infty}\left[\frac{6}{5^{n+1}}-\frac{2}{8^{n-1}}\right] = \sum_{n=1}^{\infty}\frac{6}{5^{n+1}} - \sum_{n=1}^{\infty}\frac{2}{8^{n-1}}$$

$$= \frac{3}{10} - \frac{16}{7}$$

$$= \frac{-139}{70}$$