Example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Lets look at the partial sum

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$S_n = \left[\frac{1}{1} - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \frac{1}{4}\right] + \dots$$

$$+ \left[\frac{1}{n-2} - \cancel{\frac{1}{n-1}}\right] + \left[\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right] + \left[\cancel{\frac{1}{n}} - \frac{1}{n+1}\right]$$

$$= 1 - \frac{1}{n+1}$$

Then $\displaystyle S = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} \left[1 - \frac{1}{n+1}\right]$$

$$= 1$$

Example:

$$\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = \sum_{n=1}^{\infty}\left[\frac{2}{n} - \frac{2}{n+2}\right]$$ using partial fractions.

Lets look at the partial sum in order to find the sum.

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots + a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} + a_n$$

as a rule to see how the partial sum is behaving you should write enough first and last terms so that you obtain a full block of cancellation.

$$S_n = \left[\frac{2}{1} - \cancel{\frac{2}{3}}\right] + \left[\frac{2}{2} - \cancel{\frac{2}{4}}\right] + \left[\cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}\right] + \left[\cancel{\frac{2}{4}} - \frac{2}{7}\right] + \left[\cancel{\frac{2}{5}} - \frac{2}{8}\right] + \dots +$$

$$+ \left[\frac{2}{n-4} - \cancel{\frac{2}{n-2}}\right] + \left[\frac{2}{n-3} - \cancel{\frac{2}{n-1}}\right] + \left[\cancel{\frac{2}{n-2}} - \cancel{\frac{2}{n}}\right] + \left[\cancel{\frac{2}{n-1}} - \frac{2}{n+1}\right] + \left[\cancel{\frac{2}{n}} - \frac{2}{n+2}\right]$$

$$= 2 + 1 - \frac{2}{n+1} - \frac{2}{n+2}$$

$$S = \sum_{n=1}^{\infty} \frac{4}{n(n+2)}$$

$$= \lim_{n\to\infty} S_n$$

$$= 2 + 1 + \lim_{n\to\infty} \cancel{\frac{-2}{n+1}}^{\,0} - \cancel{\frac{2}{n+2}}^{\,0}$$

$$= 3$$

Example:

$$\sum_{1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$ this is in fact a telescoping series

$$= \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln(n)\right]$$ by properties of log.

Lets look at the partial sum

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$= \left[\cancel{\ln 2} - \ln 1\right] + \left[\cancel{\ln 3} - \cancel{\ln 2}\right] + \left[\ln 4 - \cancel{\ln 3}\right] + \dots$$

$$+ \left[\ln \cancel{(n-1)} - \ln(n-2)\right] + \left[\cancel{\ln(n)} - \cancel{\ln(n-1)}\right] + \left[\ln(n+1) - \cancel{\ln(n)}\right]$$

$$= -\ln 1 + \ln(n+1)$$

$$\therefore S = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \left[-\ln 1 + \ln(n+1)\right]$$

= Does not Converge.