

Test 2

This test is graded out of 43 marks. No books, notes, no graphing calculator or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (3 marks)

Integrate the following indefinite integral:

$$\int t^2 + \frac{1}{t} + \frac{1}{t^2} dt = \frac{t^3}{3} + \ln|t| - \frac{1}{t} + C$$

Question 2. (5 marks)

Integrate the following indefinite integral:

$$\int \sec^8 3x \tan 3x dx = \int \sec^7 3x \sec 3x \tan 3x dx$$

Let
① $u = \sec 3x$

$$du = \sec 3x \tan 3x \cdot 3 dx$$

$$\textcircled{2} \frac{du}{3} = \sec 3x \tan 3x dx \quad \textcircled{1} = \int u^7 \sec 3x \tan 3x dx$$

$$\textcircled{2} = \frac{1}{3} \int u^7 du$$

$$= \frac{u^8}{8} + C$$

$$= \frac{\sec^8 3x}{8} + C$$

Question 3. (3 marks)

Integrate the following definite integral:

$$\int_1^3 \frac{x}{(2-x^2)^{2/3}} dx \quad \textcircled{1} = \int_1^3 \frac{x}{u^{2/3}} dx$$

Let

$$\textcircled{1} u = 2 - x^2$$

$$du = -2x dx$$

$$\textcircled{2} -\frac{du}{2} = x dx$$

$$\textcircled{2} = -\frac{1}{2} \int_1^{-7} \frac{du}{u^{2/3}}$$

$$= -\frac{1}{2} \left[3u^{1/3} \right]_1^{-7}$$

$$= -\frac{3}{2} (-7)^{1/3} + \frac{3}{2} (1)^{1/3}$$

$$= \frac{3\sqrt[3]{7}}{2} + \frac{3}{2}$$

$$u(1) = 2 - 1 = 1$$

$$u(3) = -7$$

Question 4. (3 marks)

Integrate the following definite integral:

$$\begin{aligned}
 \int_1^2 \frac{u-3}{\sqrt{u}} du &= \int_1^2 (u-3) u^{-\frac{1}{2}} du \\
 &= \int_1^2 u^{\frac{1}{2}} - 3u^{-\frac{1}{2}} du \\
 &= \left[\frac{2u^{\frac{3}{2}}}{3} - 2 \cdot 3u^{\frac{1}{2}} \right]_1^2 \\
 &= \frac{2(2)^{\frac{3}{2}}}{3} - 2 \cdot 3(2)^{\frac{1}{2}} - \frac{2(1)^{\frac{3}{2}}}{3} + 2 \cdot 3(1)^{\frac{1}{2}} \\
 &= \frac{2\sqrt{8}}{3} - 6\sqrt{2} - \frac{2}{3} + 6 \\
 &= \frac{4\sqrt{2}}{3} - 6\sqrt{2} - \frac{2}{3} + 6 \\
 &= \frac{16}{3} - \frac{14\sqrt{2}}{3}
 \end{aligned}$$

Question 5. Sketch (2 marks) and area (2 marks)Sketch the graph of the algebraic function $f(x) = x^2 + 2x + 1$ and $g(x) = 5x + 5$ then find the area bounded by the two functions. $f(x) = x^2 + 2x + 1 = (x+1)^2$, the vertex of the parabola is at $(-1, 0)$

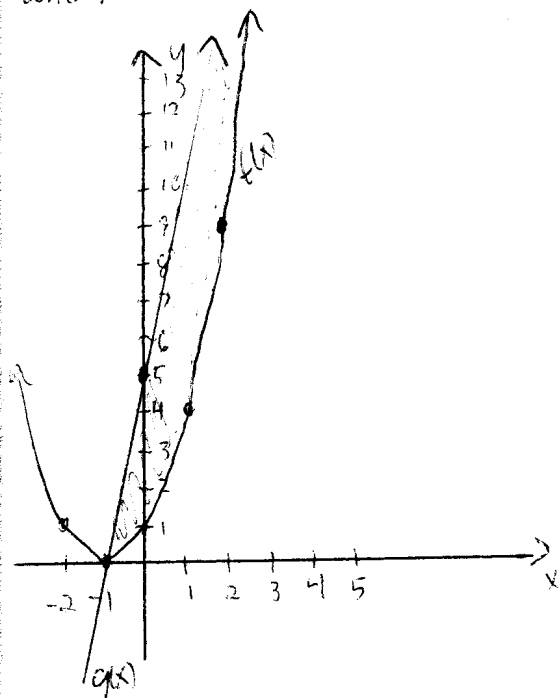
$f(x) = g(x)$

$x^2 + 2x + 1 = 5x + 5$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

∴ the curve intersect at $x = -1$ and $x = 4$.



$$\text{Area} = \int_{-1}^4 g(x) - f(x) dx$$

$$= \int_{-1}^4 5x + 5 - x^2 - 2x - 1 dx$$

$$= \int_{-1}^4 -x^2 + 3x + 4 dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x \right]_{-1}^4$$

$$= \left[-\frac{(4)^3}{3} + \frac{3(4)^2}{2} + 4(4) \right] - \left[-\frac{(-1)^3}{3} + \frac{3(-1)^2}{2} + 4(-1) \right]$$

$$= \left[-\frac{64}{3} + \frac{48}{2} + 16 \right] - \left[\frac{1}{3} + \frac{3}{2} - 4 \right]$$

$$= -\frac{64}{3} + \frac{48}{2} + 16 + \frac{13}{6}$$

$$= \frac{125}{6}$$

Question 6. (5 marks)

Integrate the following indefinite integral:

$$\int \frac{x^2}{x-2} dx$$

Let's first divide

$$x-2 \overline{) \begin{array}{r} x+2 \\ x^2+0x+0 \\ -(x^2-2x) \\ \hline 2x+0 \\ -(2x-4) \\ \hline 4 \end{array}}$$

$$= \int x+2 + \frac{4}{x-2} dx$$

$$= \frac{x^2}{2} + 2x + 4 \ln|x-2| + C$$

$$\therefore \frac{x^2}{x-2} = x+2 + \frac{4}{x-2}$$

Question 7. (5 marks)

Integrate the following indefinite integral:

$$\int e^x \cos x dx$$

$$\int u dv = uv - \int v du$$

$$u = e^x \quad du = e^x dx$$

$$v = \sin x \quad dv = \cos x dx$$

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

Apply integration by parts again

$$u = e^x \quad du = e^x dx$$

$$v = -\cos x \quad dv = \sin x dx$$

$$\int e^x \cos x dx = e^x \sin x - \left[-e^x \cos x - \int -\cos x e^x dx \right]$$

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x dx = \frac{1}{2} e^x [\sin x + \cos x] + C$$

Question 8. (5 marks)

Integrate the following indefinite integral:

$$\begin{aligned}
 \int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx \\
 &= \frac{1}{4} \int 1 - \cos^2 2x \, dx \\
 &= \frac{1}{4} \int \sin^2 2x \, dx \\
 &= \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx \\
 &= \frac{1}{8} \int 1 - \cos 4x \, dx \\
 &= \frac{1}{8} \left[x - \frac{\sin 4x}{4} \right] + C
 \end{aligned}$$

Question 9. (5 marks)

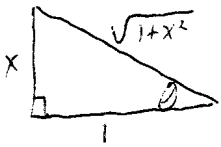
Integrate the following indefinite integral:

$$\int \frac{6x^3}{\sqrt{1+x^2}} \, dx \quad \textcircled{1} \int \frac{6(\tan^3 \theta)}{\sqrt{1+\tan^2 \theta}} \, dx$$

Let

$$\textcircled{1} x = \tan \theta$$

$$\textcircled{2} dx = \sec^2 \theta \, d\theta$$



$$\textcircled{2} = 6 \int \frac{\tan^3 \theta \sec^2 \theta}{\sqrt{\sec^2 \theta}} \, d\theta$$

$$= 6 \int \tan^3 \theta \sec \theta \, d\theta$$

$$= 6 \int \tan^2 \theta \tan \theta \sec \theta \, d\theta$$

$$= 6 \int (\sec^2 \theta - 1) \tan \theta \sec \theta \, d\theta$$

$$\textcircled{1} = 6 \int (u^2 - 1) \tan \theta \sec \theta \, d\theta$$

$$= 6 \int (u^2 - 1) \, du$$

$$= 6 \left[\frac{u^3}{3} - u \right] + C$$

$$= 2 \sec^3 \theta - 6 \sec \theta + C$$

$$= 2(\sqrt{1+x^2})^3 - 6\sqrt{1+x^2} + C$$

$$\text{Let } u = \sec \theta$$

$$du = \tan \theta \sec \theta \, d\theta$$

$$\begin{aligned}
 \sec \theta &= \frac{h}{a} \\
 &= \sqrt{1+x^2}
 \end{aligned}$$

Question 10. (5 marks)

Integrate the following indefinite integral:

$$\int \frac{3x^2 + 7x + 1}{x^3 + 2x^2 + x} dx \quad \text{lets use partial fractions} \quad \frac{3x^2 + 7x + 1}{x^3 + 2x^2 + x} = \frac{3x^2 + 7x + 1}{x(x+1)^2}$$

$$\frac{3x^2 + 7x + 1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$3x^2 + 7x + 1 = A(x+1)^2 + Bx(x+1) + Cx$$

Let $x = 0$

$$3(0)^2 + 7(0) + 1 = A(0+1)^2$$

$$1 = A$$

Let $x = -1$

$$3(-1)^2 + 7(-1) + 1 = C(-1)$$

$$3 = C$$

Let $x = 1$

$$3(1)^2 + 7(1) + 1 = 1(1+1)^2 + B(1)(1+1) + 3(1)$$

$$11 = 4 + 2B + 3$$

$$2 = B$$

$$\therefore \int \frac{1}{x} + \frac{2}{(x+1)} + \frac{3}{(x+1)^2} dx$$

$$= \ln|x| + 2\ln|x+1| - \frac{3}{(x+1)} + C$$

Bonus Question. (3 marks)

Integrate the following indefinite integral:

$$\int \frac{1}{2t \cos^2 t + t - t \cos 2t} dt = \int \frac{1}{t(2\cos^2 t + 1 - \cos 2t)} dt$$

$$= \int \frac{1}{t(2(\frac{1+\cos 2t}{2}) + 1 - \cos 2t)} dt$$

$$= \int \frac{1}{2t} dt$$

$$= \frac{1}{2} \ln|t| + C$$