**Dawson College**: **Calculus II**: 201-NYB-05-C1: Winter 2008

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# Test 3

This test is graded out of 50 marks. No books, notes, no graphing calculator or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

**Question 1.** *(5 marks)* Determine the indeterminate form, then evaluate the limit, using L'Hôpital Rule if necessary:

$\lim_{x\to\infty} \frac{\ln x^4}{x^3}$ the indeterminate form is $\frac{\infty}{\infty}$

We can therefore use L'Hôpital's Rule

$$= \lim_{x\to\infty} \frac{\frac{4}{x}}{3x^2}$$

$$= \lim_{x\to\infty} \frac{4}{3x^3}$$

$$= 0$$

**Question 2.** *(5 marks)* Determine the indeterminate form, then evaluate the limit, using L'Hôpital Rule if necessary:

$\lim_{x\to\infty} x^2 e^{-2x}$ has indeterminate form $\infty \cdot 0$. So we rewrite the limit as

$= \lim_{x\to\infty} \frac{x^2}{e^{2x}}$ has indeterminate form $\frac{\infty}{\infty}$, we use L'Hôpitals rule.

$= \lim_{x\to\infty} \frac{2x}{2e^{2x}}$ has indeterminate form $\frac{\infty}{\infty}$, we use L'Hôpitals rule.

$= \lim_{x\to\infty} \frac{2}{4e^{2x}}$

$= 0$

**Question 3.** *(5 marks)* Integrate the following improper integral if it converges:

$$\int_0^\infty xe^{-x/2}dx$$

Integration by parts.

$u = x \qquad du = dx$

$v = -2e^{-x/2} \qquad dv = e^{-x/2}dx$

$$= \lim_{b\to\infty} \int_0^b xe^{-x/2}dx$$

$$= \lim_{b\to\infty} \left[ xe^{-x/2}\Big|_0^b + 2\int_0^b e^{-x/2}dx \right]$$

$$= \lim_{b\to\infty} \left[ be^{-b/2} - 0e^{-0/2} + 2\left[-2e^{-x/2}\right]_0^b \right]$$

$$= \lim_{b\to\infty} \left[ be^{-b/2} + 4e^{-0/2} - \cancel{4e^{-b/2}}^{\to 0} \right]$$

$$= \lim_{b\to\infty} \left[ \frac{b}{e^{+b/2}} + 4 \right]$$

$$= 4 + \lim_{b\to\infty} \frac{b}{e^{b/2}}$$

has indeterminate form $\frac{\infty}{\infty}$

$$= 4 + \lim_{b\to\infty} \frac{1}{\frac{e^{b/2}}{2}}$$

$$= 4$$

**Question 4.** *(5 marks)* Integrate the following improper integral if it converges:

$$\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx$$

$$= \lim_{b \to 8} \int_0^b \frac{1}{\sqrt[3]{8-x}}\,dx$$

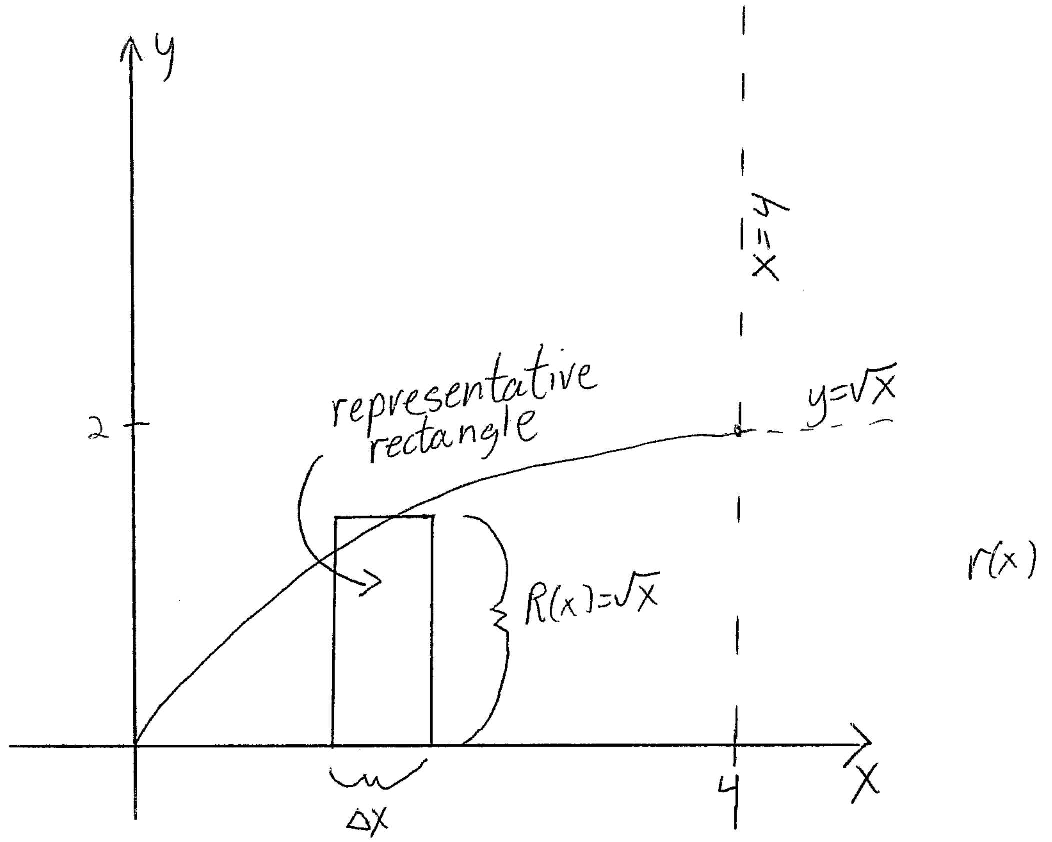
$$= \lim_{b \to 8} \left[ \frac{-3(8-x)^{\frac{2}{3}}}{2} \right]_0^b$$

$$= \lim_{b \to 8} \left[ \frac{3(8-0)^{2/3}}{2} - \frac{3(8-b)^{2/3}}{2} \right]$$

$$= \frac{3\,8^{2/3}}{2}$$

$$= 6$$

**Question 5.** *(5 marks)* Find the volume of the solid generated by revolving the region bounded by the graphs of the equations: $y = \sqrt{x}$, $y = 0$, $x = 4$ about the $x$-axis.




Representative element: $\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$

$= \pi[x]\Delta x$

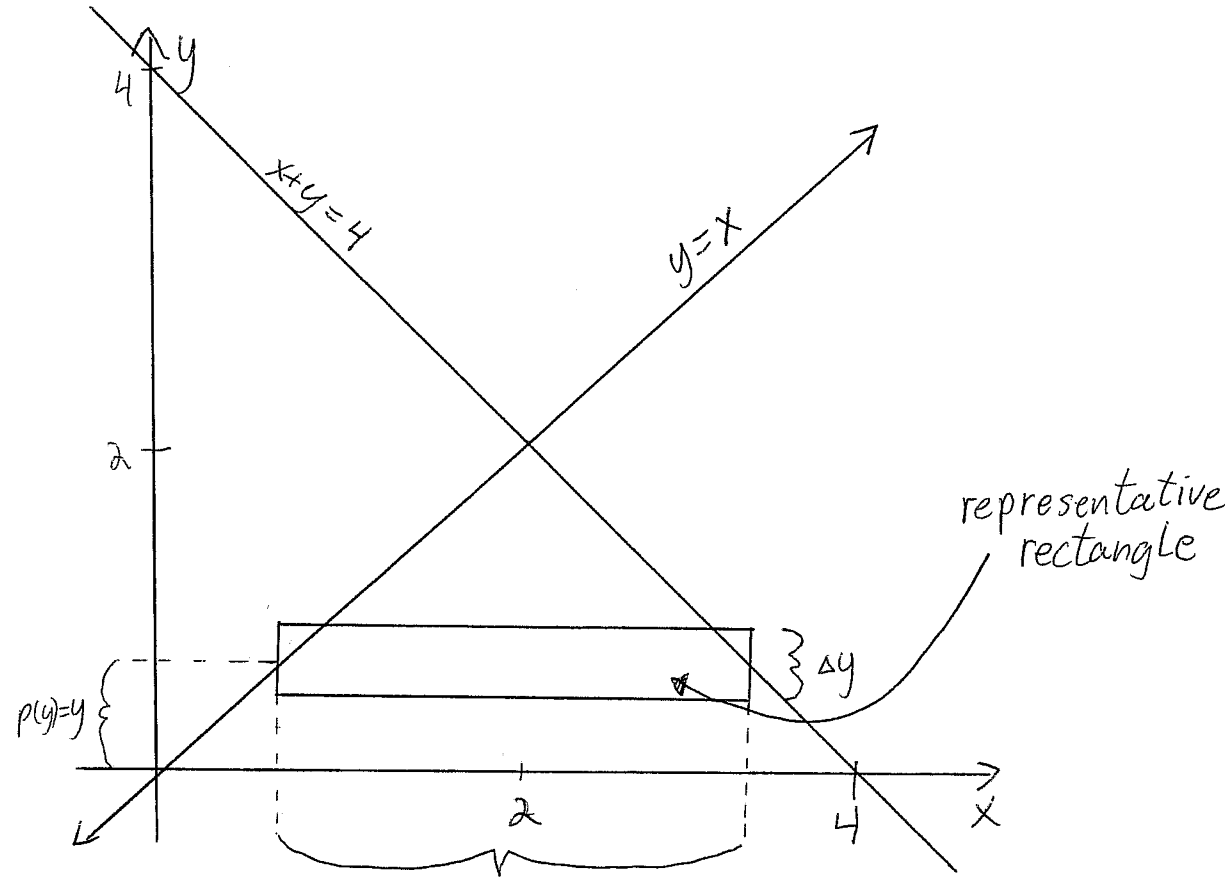
$$\int_0^4 \pi x\,dx = \left.\frac{\pi x^2}{2}\right|_0^4$$

$$= \frac{\pi(4)^2}{2}$$

$$= 8\pi$$

$\therefore$ the volume is $8\pi$

**Question 6.** *(5 marks)* Find the volume of the solid generated by revolving the region bounded by the graphs of the equations: $x+y=4$, $y=x$, $y=0$ about the $x$-axis.




$$h(y) = 4-y-y$$
$$= 4-2y$$

Representative element:

$$\Delta V = 2\pi p(y) h(y) \Delta y$$
$$\Delta V = 2\pi y (4-2y) \Delta y$$

$$\int_0^2 2\pi y(4-2y)\,dy = 2\pi \int_0^2 4y - 2y^2\,dy$$

$$= 2\pi \left[2y^2 - \frac{2y^3}{3}\right]_0^2$$

$$= 2\pi \left[2(2)^2 - \frac{2(2)^3}{3}\right]$$

$$= 2\pi \left[8 - \frac{16}{3}\right]$$

$$= 2\pi \left[\frac{24-16}{3}\right]$$

$$= \frac{16\pi}{3}$$

$\therefore$ the volume is $\frac{16\pi}{3}$

**Question 7.** *(5 marks)* Find the arc length of the graph of the function $y = \frac{x^5}{10} + \frac{1}{6x^3}$ over the interval $[1,2]$.

$$S = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{dy}{dx} = \frac{5x^4}{10} - \frac{3}{6x^4}$$

$$= \frac{x^4}{2} - \frac{1}{2x^4}$$

$$= \int_1^2 \sqrt{1 + \left(\frac{x^4}{2} - \frac{1}{2x^4}\right)^2}$$

$$= \int_1^2 \sqrt{1 + \frac{x^8}{4} - \frac{1}{2} + \frac{1}{4x^8}} \, dx$$

$$= \int_1^2 \sqrt{\frac{x^8}{4} + \frac{1}{2} + \frac{1}{4x^8}} \, dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^4}{2} + \frac{1}{2x^4}\right)^2} \, dx$$

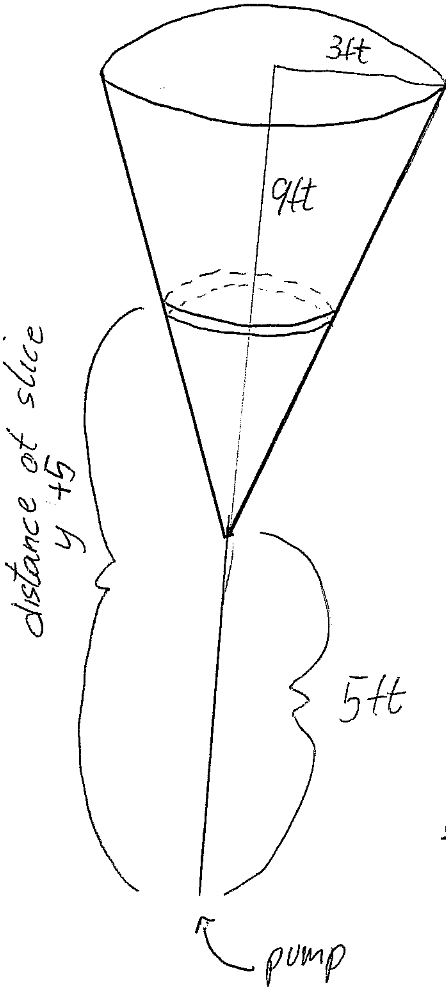
$$= \int_1^2 \frac{x^4}{2} + \frac{1}{2x^4} \, dx$$

$$= \left[\frac{x^5}{10} - \frac{x^{-3}}{6}\right]_1^2$$

$$= \left[\frac{2^5}{10} - \frac{2^{-3}}{6} - \frac{1^5}{10} + \frac{1^{-3}}{6}\right]$$

$$= \left[\frac{32}{10} - \frac{1}{48} - \frac{1}{10} + \frac{1}{6}\right] = \frac{779}{240}$$

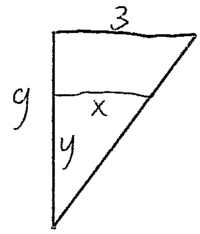
**Question 8.** *(5 marks)* An open tank has the shape of a circular cone with its tip oriented downward. The tank is 6 feet across the top and 9 feet high. How much work is done in filling the tank by pumping the water from 5 feet below the tank? (The water weighs 62.4 pounds per cubic feet)




Volume of Slice:

$$\Delta V = \pi x^2 \Delta y$$

but we want the volume of the slice with respect to y. Therefore we set up the following ratio




$$\frac{x}{y} = \frac{3}{9}$$

$$x = \frac{y}{3}$$

$$\therefore \Delta V = \pi \left(\tfrac{1}{3}y\right)^2 \Delta y$$

$$\Delta V = \pi \frac{y^2}{9} \Delta y$$

Force excerted by Slice:

$$\Delta F = \Delta V (\text{weigh of water per cubic feet})$$

$$\Delta F = 62.4 \left(\pi \frac{y^2}{9}\right) \Delta y$$

$$\Delta F = \frac{104\pi}{15} y^2 \Delta y$$

Work to move Slice: $\Delta W = \Delta F \times \text{distance}$

$$\Delta W = \frac{104\pi}{15} y^2 (y+5) \Delta y$$

$$\text{Work} = \int_0^9 \frac{104\pi}{15}(y^3 + 5y^2)\,dy = \left[\frac{104\pi}{15}\left(\frac{y^4}{4} + \frac{5y^3}{3}\right)\right]_0^9$$

$$= \left[\frac{104\pi}{15}\left(\frac{9^4}{4} + \frac{5(9)^3}{3}\right)\right]$$

$$= \left[\frac{104\pi}{15}\left(\frac{11421}{4}\right)\right] = \frac{98982\pi}{5} \doteq 62192 \text{ lb}\cdot\text{ft}$$

**Question 9.** *(5 marks)* Integrate the following indefinite integral:

$$\int \frac{x^2-1}{x^3+x}\,dx$$

$$= \int \frac{x^2-1}{x(x^2+1)}\,dx$$ integration using partial fractions

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2-1 = A(x^2+1) + (Bx+C)x$$

Lets solve for $A, B, C$.

Let $x=0$

$$(0)^2-1 = A(0^2+1) + (B(0)+C)(0)$$

$$-1 = A$$

Let $x=1$

$$(1)^2-1 = A(1^2+1) + (B(1)+C)(1)$$

$$0 = 2A + B + C$$

$$0 = -2 + B + C$$

$$2-B = C \quad (1)$$

Let $x=-1$

$$(-1)^2-1 = A((-1)^2+1) + (B(-1)+C)(-1)$$

$$0 = 2A + B - C$$

$$2 = B - C$$

$$2 = B - 2 + B \quad \text{by } (1)$$

$$4 = 2B$$

$$2 = B$$

$\therefore$ by (1) $C = 0$

$$= \int \frac{-1}{x} + \frac{2x}{x^2+1}\,dx$$

$$= -\ln|x| + \ln|x^2+1| + C$$

**Question 10.** *(5 marks)* Integrate the following indefinite integral:

$$\int x\sqrt{1+x^2}\,dx$$

Let ① $x = \tan\theta$

② $dx = \sec^2\theta\, d\theta$

$$\overset{①}{=} \int \tan\theta\sqrt{1+\tan^2\theta}\,dx$$

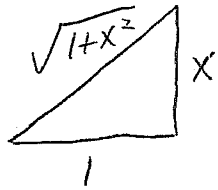
$$\overset{②}{=} \int \tan\theta\sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta$$

$$= \int \tan\theta\ \sec^3\theta\, d\theta$$

$$= \int \sec^2\theta\ \sec\theta\tan\theta\, d\theta$$

$$= \frac{\sec^3\theta}{3} + C$$

$$= \frac{\left(\sqrt{1+x^2}\right)^3}{3} + C$$




$$\sec\theta = \frac{h}{a} = \sqrt{1+x^2}$$

**Bonus Question.** *(3 marks)* Evaluate the following limit:

$$\lim_{x\to 1^+} (\ln x)^{x-1}$$

See other version