Dawson College: **Calculus 1**: 201-NYA-05-C2: Winter 2008: Th. April 3rd 2008

Name: ____________
Student ID: SOLUTIONS

TEST 2

This test is graded out of 60 marks (there are 5 bonus marks available). No books or notes are allowed. SHOW ALL YOUR WORK. If you need more space for your answer use the back of the page.

Question 1. *(4 marks each)*
Find the derivative.
(a) $f(x) = \frac{\ln x}{\sin x}$

quotient rule

$$f'(x) = \frac{(1/x)\sin x - (\cos x)\ln x}{\sin^2 x}$$

$$\boxed{= \frac{\sin x - x\cos x \ln x}{x \sin^2 x}}$$

(b) $f(x) = (4 - e^{\frac{x}{5}})^5$.

CHAIN rule + CHAIN RULE

$$f'(x) = 5(4-e^{x/5})^4 \frac{d}{dx}(4-e^{x/5})$$

$$= 5(4-e^{x/5})^4\left(-e^{x/5}\cdot\frac{1}{5}\right)$$

$$\boxed{= -e^{x/5}(4-e^{x/5})^4}$$

Question 2. *(5 marks each)*
Find the derivative.
(a) $f(x) = \sin(x^2 + \cos x)$

CHAIN RULE

$$\boxed{f'(x) = \cos(x^2 + \cos x)\,(2x - \sin x)}$$

(b) $f(x) = e^x \left(\frac{x}{\sin x}\right)$

PRODUCT + QUOTIENT RULES

$$f'(x) = e^x\left(\frac{x}{\sin x}\right) - \left(\frac{\sin x - x\cos x}{\sin^2 x}\right)e^x$$

$$= \frac{e^x}{\sin x}\left(x - \left(\frac{\sin x - x\cos}{\sin x}\right)\right)$$

$$\boxed{= \frac{e^x}{\sin x}\left(x - 1 + x\,\mathrm{cotan}\,x\right)}$$

(c) $f(x) = x^2 \sec x$

METHOD 1 PRODUCT RULE

$$f'(x) = 2x\sec x + (\sec x \tan x)x^2$$

$$\boxed{= x\sec x(2 + x\tan x)}$$

Method 2 quotient RULE

$$f(x) = x^2 \sec x = \frac{x^2}{\cos x}$$

$$f'(x) = \frac{2x\cos x - (-\sin x)x^2}{\cos^2 x} = \boxed{\frac{x(2\cos x + x\sin x)}{\cos^2 x}}$$

(d) $f(x) = \frac{-2}{-5x^{\frac{5}{2}}}$

REWRITE $f(x) = \frac{-2}{-5x^{5/2}} = \frac{2}{5}x^{-5/2}$

$$f'(x) = \frac{2}{5}\left(-\frac{5}{2}x^{-7/2}\right)$$

$$= -x^{-7/2}$$

$$\boxed{= -\frac{1}{x^{7/2}}}$$

Question 3. *(5 marks)*
Find the equation of the tangent line to the curve $y = \sqrt{x^2 + \sin x}$ at the point (π, π).

y' At (π, π) gives the slope of the tangent

$$y' = \frac{1}{2}(x^2 + \sin x)^{-1/2}(2x + \cos x)$$

At $x = \pi$
$$y' = \frac{1}{2}(\pi^2 + \sin\pi)^{-1/2}(2\pi + \cos\pi)$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{\pi^2}}\right)(2\pi - 1)$$
$$= \frac{2\pi - 1}{2\pi}$$

The equation is $y = \left(\frac{2\pi-1}{2\pi}\right)x + b$

substitute (π, π) $\Longrightarrow$ $\pi = \left(\frac{2\pi-1}{2\cancel{\pi}}\right)\cancel{\pi} + b$

$b = \pi - \left(\frac{2\pi - 1}{2}\right) = \frac{1}{2}$

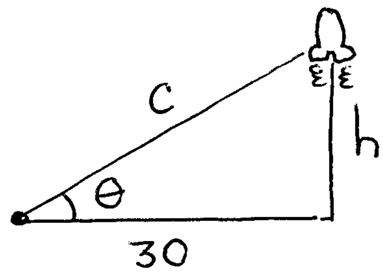
so $\boxed{y = \left(\frac{2\pi-1}{2\pi}\right)x + \frac{1}{2}}$

Question 4. *(5 marks)*
Find the derivative of $y = (\cos x)^{2x}$. (Hint: Use logarithmic differentiation)

$$\ln y = \ln(\cos x)^{2x}$$
$$\ln y = 2x \ln\cos x$$
$$\frac{1}{y}y' = 2\ln\cos x + \frac{1}{\cos x}(-\sin x)2x$$
$$y' = y\left[2\ln\cos x - 2x\tan x\right]$$
$$y' = (\cos x)^{2x}\left[2\ln\cos x - 2x\tan x\right]$$

Question 5 *(10 marks)*
A television camera is positioned 30m from the base of a rocket launching pad. A rocket rises vertically and its speed is 75m/s at the moment it is 40m in the air.
(a) How fast is the distance from the camera to the rocket changing at the moment the rocket is 40m in the air?
(b) If the television camera is always kept focused on the rocket, how fast is the camera's angle of elevation changing at the moment the rocket is 40m in the air?

SKETCH



Given
$\frac{dh}{dt} = 75\ \text{m/s}$
when $h = 40\text{m}$

(a) Looking for dc/dt when $h=40$

$$30^2 + h^2 = c^2$$

$$2h\frac{dh}{dt} = 2c\frac{dc}{dt} \Rightarrow 2(40)(75) = 2c\frac{dc}{dt}$$

$$\Rightarrow 6000 = 2c\frac{dc}{dt}$$

When $h=40$ $c^2 = 30^2+40^2$
so $c = 50$ Thus $\frac{dc}{dt} = \frac{6000}{100} = \frac{600}{10}$

$$\boxed{= 60\ \text{m/s}}$$

(b) Looking for $d\theta/dt$ when $h=40$

$$\tan\theta = \frac{h}{30} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dh}{dt}$$

$\cos\theta = 30/50 = 3/5$ so $\frac{d\theta}{dt} = \left(\frac{3}{5}\right)^2\left(\frac{1}{30}\right)(75)$

$$= \boxed{9/10\ \text{rad/s}}$$

Question 6. *(10 marks)*
Use implicit differentiation to find dy/dx (y') and d^2y/dx^2 (y") for the curve $\cos y = x^2 y$. Express your answer in terms of x and y only.

$$\cos y = x^2 y$$

$$(-\sin y)y' = 2xy + y'x^2$$

$$y'(-\sin y - x^2) = 2xy$$

$$\boxed{y' = \frac{2xy}{-x^2-\sin y}}$$

$$y'' = \frac{(2y+2xy')(-x^2-\sin y) - (-2x-\cos y\, y')2xy}{(-x^2-\sin y)^2}$$

substitute $y' = \dfrac{2xy}{(-x^2-\sin y)^2}$

$$\boxed{y'' = \frac{\left(2y+2x\left(\frac{2xy}{-x^2-\sin y}\right)\right)(-x^2-\sin y) - \left(-2x-\cos y\left(\frac{2xy}{-x^2-\sin y}\right)\right)2xy}{(-x^2-\sin y)^2}}$$

BONUS QUESTION *(5 marks)*
Prove the rule of differentiation $\frac{d}{dx}[tanx] = \sec^2 x$, using trigonometric identities and the quotient rule.

$$\tan x = \frac{\sin x}{\cos x} \quad \& \quad \sec^2 x = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{\cos x(\cos x) - (-\sin x)(\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$