Compound Interest

March 2, 2009

Compound Interest:

- *Compound interest* is a type of interest where after a defined period of time the initial principal along with the interest is reinvested. The process of reinvesting is called *compounding*.

- **Example:** (Compound interest) What is the future value of $\$1\ 000$ at 10% compounded yearly for 3 years.

Year	Principal	Interest	Future Value
1	$1 000	$100	$1 100
2	$1 100	$110	$1 210
3	$1 210	$121	$1 331

- **Example:** (Simple interest) What is the future value of $\$1\ 000$ at 10% for 3 years.

$$
\begin{aligned}
S &= P(1+rt) \\
&= 1000(1+0.1(3)) \\
&= \$1300
\end{aligned}
$$

Compound Interest: Future Value

- The future value formula for compound interest is given by

  $$FV = PV(1+i)^n$$

  where FV is the future value, PV is the principal value (or present value), i the *periodic rate of interest*, n is the number of compounding period.

- The value for i and n can be determined by the two following formula.

  $$\begin{aligned} i &= \frac{j}{m} \\ n &= tm \end{aligned}$$

  where j is the *nominal interest rate* (i.e. interest rate per year), m is the number of compounding per year, t is the length of the term of investment or loan (the unit being years).

Compound Interest: Future Value

- **Compounding Frequency Table:**

Compounding Frequency	**Lenght of Compounding Period**	m
Annually	12 months	1
Semi-Annually	6 months	2
Quarterly	3 months	4
Monthly	1 month	12
Daily	1 day	365

Compound Interest: Future Value: Example

- What is the future value of $\$8\,000$ invested at 3.75% per annum, compounded daily for 3 years?

$$\begin{aligned} n &= mt \\ &= 365(3) \\ &= 1095 \end{aligned}$$

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{0.0375}{365} \\ &= 0.000102739 \end{aligned}$$

$$\begin{aligned} FV &= PV(1+i)^n \\ &= 8000(1+0.000102739)^{1095} \\ &= \$8952.53 \end{aligned}$$

Compound Interest: Future Value: Example:

- Find the accumulated value of $\$2\,593.23$ invested for $2\frac{1}{3}$ years at a nominal rate of 6% compounded quarterly?

$$\begin{aligned} n &= mt \\ &= 4\left(2\frac{1}{3}\right) \\ &= 9\frac{1}{3} \\ &= \frac{28}{3} \end{aligned}$$

$$\begin{aligned} i &= \frac{j}{m} \\ &= \frac{0.06}{4} \\ &= 0.015 \end{aligned}$$

$$\begin{aligned} FV &= PV(1+i)^n \\ &= 2593.23(1+0.015)^{\frac{28}{3}} \\ &= \$2979.83 \end{aligned}$$

Compound Interest: Present Value

- *Present value* is the required principal needed to obtain a future value. The equation is obtained from the future value equation $FV = PV(1+i)^n$ by isolating PV:

$$\begin{aligned} PV &= \frac{FV}{(1+i)^n} \\ PV &= FV(1+i)^{-n} \end{aligned}$$

- **Example:** What principal is required to obtain a future value of $\$1\,520$ at a nominal rate of 5% compounded monthly for 13 months?

$$n = mt = 12\left(\frac{13}{12}\right) = 13$$

$$i = \frac{j}{m} = \frac{0.05}{12} = 0.004166666$$

$$\begin{aligned} PV &= FV(1+i)^{-n} \\ &= 1520(1+0.004166666)^{-13} \\ &= \$1440.02 \end{aligned}$$

Compound Interest: Present Value: Example

- Sasha wants to have $\$25\,000$ in ten and a half years. If he can invest into an account with a nominal rate of 3.75% compounded annually, how much does Sasha need to invest now?

$$
\begin{aligned}
n &= mt \\
&= 1\left(10\frac{1}{2}\right) \\
&= \frac{21}{2}
\end{aligned}
$$

$$
\begin{aligned}
i &= \frac{j}{m} \\
&= \frac{0.0375}{1} \\
&= 0.0375
\end{aligned}
$$

$$
\begin{aligned}
PV &= FV(1+i)^{-n} \\
&= 25000(1+0.0375)^{-\frac{21}{2}} \\
&= \$16\,984.97
\end{aligned}
$$

Compound Interest: Effective Interest Rates

- *Effective rate of interest* is the equivalent interest rate which compounded annually will result in the same amount of interest as a nominal interest rate compounded more than once.

- It is used to compare different nominal interest rates with different compounding period (i.e. m).

- Let f be the effective rate of interest. The formula for f is obtained by letting the future value of PV at a rate of f compounded annually for a year equal the future value of PV at a periodic interest rate of i compounded $m > 1$ times.

$$\begin{aligned} PV(1+f)^1 &= PV(1+i)^m \\ 1+f &= (1+i)^m \\ f &= (1+i)^m - 1 \end{aligned}$$

Compound Interest: Effective Interest Rates: Example

- Find the effective rate of interest of 7% compounded: a) quartely b) monthly c) daily.

$$\begin{aligned} f &= (1+i)^m - 1 \\ &= \left(1+\frac{7\%}{4}\right)^4 - 1 \\ &= 7.19\% \end{aligned}$$

$$\begin{aligned} f &= (1+i)^m - 1 \\ &= \left(1+\frac{7\%}{12}\right)^{12} - 1 \\ &= 7.23\% \end{aligned}$$

$$\begin{aligned} f &= (1+i)^m - 1 \\ &= \left(1+\frac{7\%}{365}\right)^{365} - 1 \\ &= 7.25\% \end{aligned}$$

Compound Interest: Effective Interest Rates: Example

- If $\$2\,000$ accumulates to $\$2\,374.32$ in three and a quarter years, what is the effective annual rate?

$$
\begin{aligned}
PV(1+f)^t &= PV(1+i)^{mt} \\
2000(1+f)^{13/4} &= 2374.32 \\
(1+f)^{13/4} &= 1.18716 \\
(1+f)^{13/4} &= 1.18716 \\
1+f &= (1.18716)^{4/13} \\
f &= (1.18716)^{4/13}-1 \\
f &= 5.4\%
\end{aligned}
$$