Dawson College: **Calculus II**: 201-NYB-05 12

April 7, 2010

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 2

Question 1. *(5 marks)* Evaluate the following integral:

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2x\right) dx = \frac{1}{4}\int 1 - \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{\cos 4x}{2}\right) dx = \frac{1}{8}\int 1 - \cos 4x \, dx$$

$$= \frac{1}{8}\left(\theta - \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{1}{8}\theta - \frac{1}{32}\sin 4x + C$$

Question 2. *(5 marks)* Evaluate the following integral:

$$\int_0^{\pi/3} \sin^3 x \cos^2 x \, dx = \int_0^{\pi/3} \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int_0^{\pi/3} (1-\cos^2 x)\cos^2 x \sin x \, dx$$

LET $u = \cos x$

$du = -\sin x \, dx$

IF $x = 0 \Rightarrow u = \cos 0 = 1$

IF $x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$= \int_1^{1/2} (1-u^2)\, u^2 \sin x \frac{du}{-\sin x}$$

$$= \int_1^{1/2} (u^4 - u^2)\, du = \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{1/2}$$

$$= \left[\frac{(1/2)^5}{5} - \frac{(1/2)^3}{3}\right] - \left[\frac{1^5}{5} - \frac{1^3}{3}\right]$$

$$= \frac{1}{160} - \frac{1}{24} - \frac{1}{5} + \frac{1}{3}$$

$$= \frac{47}{480}$$

Question 3. *(8 marks)* Use a trigonometric substitution to evaluate the following integral:

$$\int x^3\sqrt{x^2+4}\,dx$$

LET $x = 2\tan\theta$ ON $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

THEN $dx = 2\sec^2\theta\,d\theta$

$\Rightarrow \sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sqrt{\tan^2\theta+1}$

$= 2\sqrt{\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$

(SINCE $\sec\theta \geq 0$ ON $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$)

$$= \int (2\tan\theta)^3\, 2\sec\theta\, 2\sec^2\theta\, d\theta$$

$$= 32\int \tan^3\theta\sec^3\theta\, d\theta$$

LET $u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$$= 32\int \tan^2\theta\sec^2\theta\sec\theta\tan\theta\, d\theta$$

$$= 32\int (\sec^2\theta - 1)\sec^2\theta\sec\theta\tan\theta\, d\theta$$

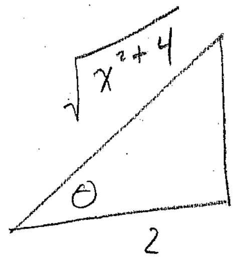
$$= 32\int (u^2-1)u^2\, du$$

$$= 32\int (u^4-u^2)\, du$$

$$= 32\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= \frac{32}{5}\sec^5\theta - \frac{32}{3}\sec^3\theta + C$$

$\tan\theta = \frac{x}{2}$



$\therefore \sec\theta = \frac{\sqrt{x^2+4}}{2}$

$$= \frac{32}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 + C$$

$$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

Question 4. *(5 marks)* Evaluate the following integral:

LET $I = \int \frac{4x^2+5x+4}{x^3+4x^2+4x}\,dx$

$$x^3+4x^2+4x$$
$$= x(x^2+4x+4)$$
$$= x(x+2)(x+2)$$

$$\frac{4x^2+5x+4}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Rightarrow \quad 4x^2+5x+4 = A(x+2)^2 + Bx(x+2) + Cx$$

LET $\underline{x=0}$

$4 = 4A$

$\boxed{1 = A}$

LET $\underline{x=-2}$

$16 = -2C$

$\boxed{-8 = C}$

LET $\underline{x=1}$

$13 = 9A + 3B + C$

$= 9 + 3B - 5$

$9 = 3B$

$\boxed{B = 3}$

$$\therefore \quad I = \int \frac{1}{x} + \frac{3}{x+2} - \frac{5}{(x+2)^2}\,dx$$

$$= \ln|x| + 3\ln|x+2| + \frac{5}{x+2} + C$$

Question 5. *(7 marks)* Evaluate the following integral:

LET $I = \int \frac{2x^4+2x^3-4x^2+2x-3}{x(x^2+1)^2}\,dx$

$$\frac{2x^4+2x^3-4x^2+2x-3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow 2x^4+2x^3-4x^2+2x-3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + (Bx+C)(x^3+x) + Dx^2+Ex$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x^2 + A$$

$\therefore$ $\underline{A=-3}$, $-3+B=2 \Rightarrow \underline{B=5}$, $\underline{C=2}$, $2(-3)+5+D=-4$

$\therefore \underline{D=-3}$

$2+E=2 \Rightarrow \underline{E=0}$

$$\therefore I = \int -\frac{3}{x} + \frac{5x+2}{x^2+1} - \frac{3x}{(x^2+1)^2}\,dx$$

$$= \int -\frac{3}{x} + \frac{5x}{x^2+1} + \frac{2}{x^2+1} - \frac{3x}{(x^2+1)^2}\,dx$$

$$= -3\ln|x| + \frac{5}{2}\ln(x^2+1) + 2\arctan x + \frac{3}{2}\cdot\frac{1}{x^2+1} + C$$

Question 6. *(6 marks)* Use Simpson's rule with $n = 6$ to approximate the following integral. Compare this with the exact value of the integral by evaluating directly.

$$\int_1^3 \frac{\ln x}{x}\, dx \qquad \Delta x = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3},$$

$x_0 = 1$, $x_1 = 1 + \frac{1}{3} = \frac{4}{3}$, $x_2 = 1 + \frac{2}{3} = \frac{5}{3}$, $x_3 = 1 + \frac{3}{3} = 2$

$x_4 = 1 + \frac{4}{3} = \frac{7}{3}$, $x_5 = 1 + \frac{5}{3} = \frac{8}{3}$, $x_6 = 1 + \frac{6}{3} = 3$

$$\therefore \int_1^3 \frac{\ln x}{x}\, dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$= \frac{1/3}{3}\left[\frac{\ln(1)}{(1)} + \frac{4\ln(4/3)}{4/3} + \frac{2\ln(5/3)}{5/3} + 4\frac{\ln(2)}{2} + \frac{2\ln(\frac{7}{3})}{\frac{7}{3}} + \frac{4\ln(\frac{8}{3})}{8/3} + \frac{\ln(3)}{3}\right]$$

$$= \frac{1}{9}\left[0 + .8630462174 + 0.6129907485 + 1.386294361 + 0.7262553089 + 1.47124388 + 0.3662040962\right]$$

$$= \frac{1}{9}\left[5.426034612\right] = 0.6028927347$$

Now

$$\int_1^3 \frac{\ln x}{x}\, dx = \int_0^{\ln 3} u\, du$$

LET $u = \ln x$

$du = \frac{1}{x}\, dx$

$x = 1 \Rightarrow u = 0$

$x = 3 \Rightarrow u = \ln 3$

$$= \left[\frac{u^2}{2}\right]_0^{\ln 3} = \frac{(\ln 3)^2}{2}$$

$$= 0.603474480 4$$

$\therefore$ |ACTUAL − APPROXIMATION| = 0.00058175

Question 7. *(5 marks)* Evaluate the following limit:

$$\lim_{x\to\infty} x\sin(1/x) = \text{“}\infty \cdot 0\text{”} = \lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}}$$

$$= \text{“}\frac{0}{0}\text{”} \overset{H}{=} \lim_{x\to\infty} \frac{\cos(1/x)\cdot 1/x^2}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\cos(1/x)$$

$$= -1$$

Question 8. *(6 marks)* Evaluate the following limit:

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \text{"} \infty - \infty \text{"}$$

$$= \lim_{x\to 0^+}\left(\frac{e^x-1}{x(e^x-1)} - \frac{x}{x(e^x-1)}\right) = \lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right)$$

$$= \text{"}\frac{0}{0}\text{"} \overset{H}{=} \lim_{x\to 0^+}\frac{e^x-1}{e^x-1+x(e^x)} = \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+}\frac{e^x}{e^x+e^x+x(e^x)} = \frac{1}{1+1+0} = \frac{1}{2}$$

Bonus. *(4 marks)* If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x(x^2+1)^3}\,dx$$

is a rational function (that is, does not have any logarithms or inverse trigonometric functions) find the value of $f'(0)$.