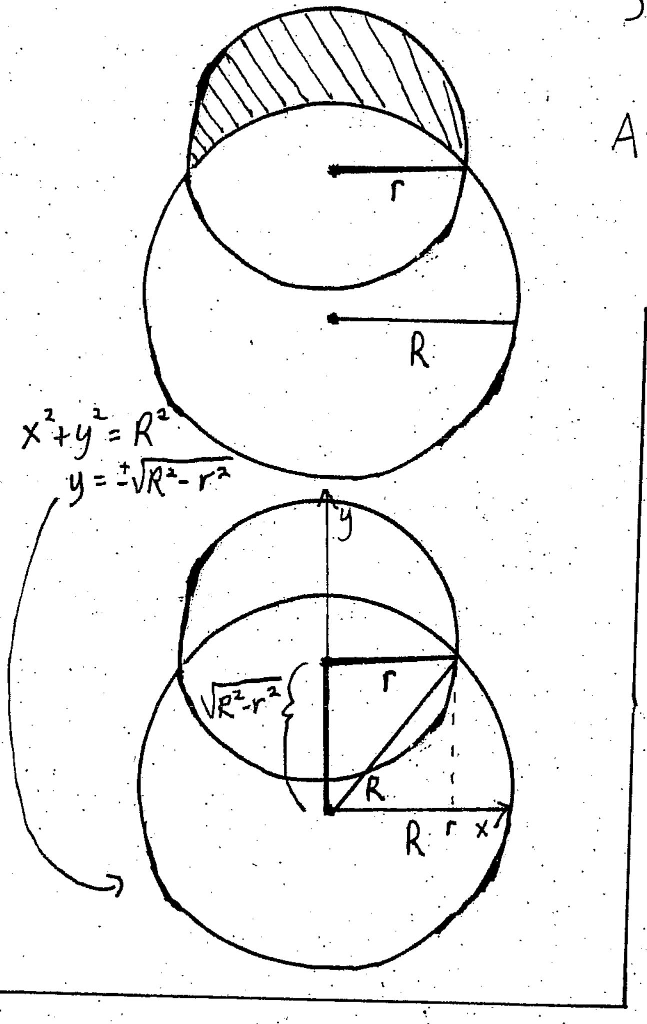
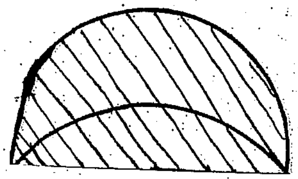
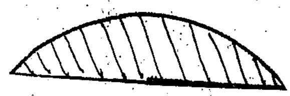
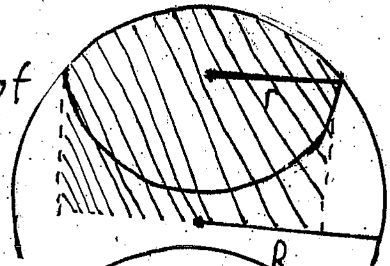
Find the area:

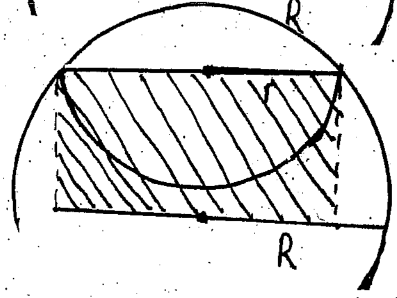


So

Area = Area of 

$-$ Area of 

$= \dfrac{\pi r^2}{2} - \Big[$ Area of 

$-$ Area of  $\Big]$

$$= \frac{\pi r^2}{2} - \left[\int_{-r}^{r} \sqrt{R^2-x^2}\,dx - 2r\sqrt{R^2-r^2} \right]$$

$$= \frac{\pi r^2}{2} - \left[R^2 \arcsin\frac{r}{R} + r\sqrt{R^2-r^2} - 2r\sqrt{R^2-r^2} \right]$$

$$= \frac{\pi r^2}{2} + r\sqrt{R^2-r^2} - R^2 \arcsin\frac{r}{R}$$

$$\int_{-r}^{r} \sqrt{R^2 - x^2}\, dx = \int_a^b \sqrt{R^2 - (R\sin\theta)^2}\, R\cos\theta\, d\theta$$

$x = R\sin\theta$

$dx = R\cos\theta\, d\theta$

$$= \int_a^b \sqrt{R^2 - R^2\sin^2\theta}\, R\cos\theta\, d\theta$$

$r = R\sin\theta$

$\frac{r}{R} = \sin\theta$

$\theta = \arcsin\frac{r}{R} = b$

$$= \int_a^b \sqrt{R^2(1-\sin^2\theta)}\, R\cos\theta\, d\theta$$

$-r = R\sin\theta$

$\frac{-r}{R} = \sin\theta$

$\theta = \arcsin\frac{-r}{R} = a$

$$= \int_a^b \sqrt{R^2\cos^2\theta}\, R\cos\theta\, d\theta$$

$$= \int_a^b R\cos\theta\, R\cos\theta\, d\theta$$

$$= \int_a^b R^2\cos^2\theta\, d\theta$$

$$= R^2\int_a^b \cos^2\theta\, d\theta$$

$$= R^2\int_a^b \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{R^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_a^b$$

$$= \frac{R^2}{2}\left[\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right]_a^b$$

$$= \frac{R^2}{2}\left[\left[b + \sin b\cos b\right] - \left[a + \sin a\cos a\right]\right]$$

$$= \frac{R^2}{2}\left[\left[\arcsin\frac{r}{R} + \sin\left(\arcsin\frac{r}{R}\right)\cos\left(\arcsin\frac{r}{R}\right)\right]\right.$$

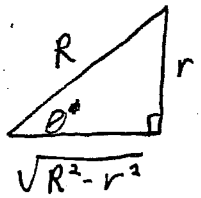
$$\left. - \left[\arcsin\frac{-r}{R} + \sin\left(\arcsin\frac{-r}{R}\right)\cos\left(\arcsin\frac{-r}{R}\right)\right]\right]$$

note:

$$\theta^* = \arcsin \frac{r}{R}$$

$$\sin\theta^* = \frac{r}{R}$$

So

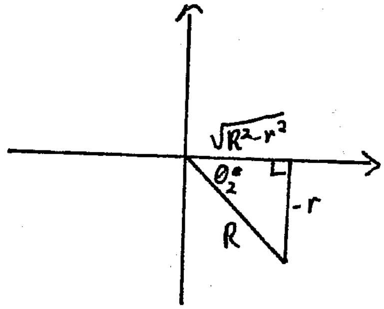


$$\cos\theta^* = \frac{\sqrt{R^2-r^2}}{R}$$

note:

$$\theta_2^* = \arcsin \frac{-r}{R}$$

$$\sin\theta_2^* = \frac{-r}{R}$$



$$\cos\theta_2^* = \frac{\sqrt{R^2-r^2}}{R}$$

note:

The inverse of an odd function is also odd

$$\therefore \arcsin\frac{-r}{R} = -\arcsin\frac{r}{R}$$

$$= \frac{R^2}{2}\left[\left[\arcsin\frac{r}{R} + \frac{r}{R}\frac{\sqrt{R^2-r^2}}{R}\right] - \left[-\arcsin\frac{r}{R} - \frac{r}{R}\frac{\sqrt{R^2-r^2}}{R}\right]\right]$$

$$= R^2\arcsin\frac{r}{R} + r\sqrt{R^2-r^2}$$