Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S4: Winter 2011

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Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate using the definition of the definite integral

$$\int_2^3 -6x^2+4x-2\,dx.$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } x_i = a + i\Delta x = 2 + \frac{i}{n} \qquad \Delta x = \frac{b-a}{n} = \frac{3-2}{n} = \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+\frac{i}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[-6\left(2+\frac{i}{n}\right)^2 + 4\left(2+\frac{i}{n}\right) - 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[-6\left[4+\frac{4i}{n}+\frac{i^2}{n^2}\right] + 8 + \frac{4i}{n} - 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[-24 - 24\frac{i}{n} - \frac{6i^2}{n^2} + 8 + 4\frac{i}{n} - 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[-18 - \frac{20i}{n} - \frac{6i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} -18 - \frac{20}{n}\sum_{i=1}^{n} i - \frac{6}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[-18n - \frac{\cancel{20}^{10}}{\cancel{n}}\frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{\cancel{6}}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{-18\cancel{n}}{\cancel{n}} - 10\frac{(n+1)}{n} - \frac{(n+1)}{n}\frac{(2n+1)}{n}\right]$$

$$= -18 - 10 - 1\cdot 2$$

$$= -30$$

Question 2. *(5 marks)* Evaluate the definite integral:

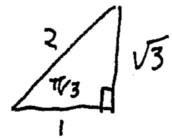
$$\int_{\pi/4}^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sin\theta\,\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \Big[-\cos\theta\Big]_{\pi/4}^{\pi/3}$$

$$= -\cos\frac{\pi}{3} + \cos\frac{\pi}{4}$$

$$= -\frac{1}{2} + \frac{1}{\sqrt{2}}$$



Question 3. *(5 marks)* Evaluate the definite integral:

$$\int_1^2 (z^2+1)\sqrt[3]{z-1}\,dz = \int_0^1 \left[(u+1)^2+1\right]\sqrt[3]{u}\,du$$

$u = z-1$
$du = dz$
$u(1) = 0$
$u(2) = 2-1 = 1$
$u+1 = z$

$$= \int_0^1 \left[u^2 + 2u + 1 + 1\right] u^{1/3}\,du$$

$$= \int_0^1 u^{7/3} + 2u^{4/3} + 2u^{1/3}\,du$$

$$= \left[\frac{3u^{10/3}}{10} + 2\cdot\frac{3}{7}u^{7/3} + 2\frac{3}{4}u^{4/3}\right]_0^1$$

$$= \frac{3}{10} + \frac{6}{7} + \frac{3}{2}$$

$$= \frac{21}{70} + \frac{60}{70} + \frac{105}{70}$$

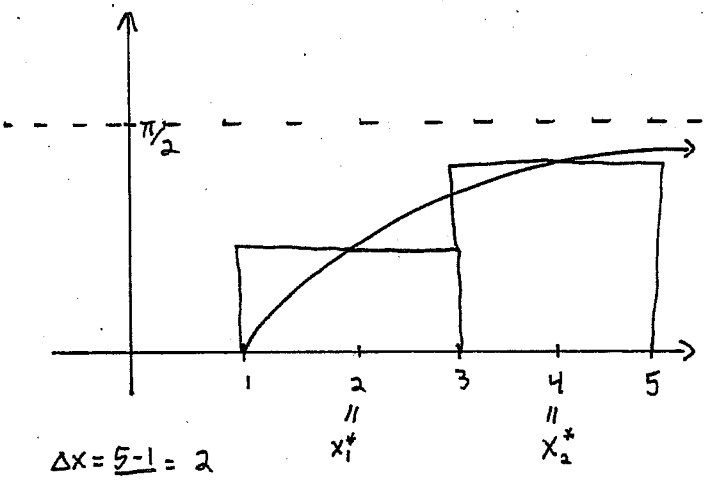
$$= \frac{186}{70}$$

$$= \frac{93}{35}$$

Question 4. *(5 marks)* Estimate the average value of the function

$$f(x) = \operatorname{arcsec}(x)$$

on the interval $[1,5]$ using two rectangles and the Midpoint Rule. Sketch the curve and approximating rectangles.



$\Delta x = \frac{5-1}{2} = 2$

Avg. val. of func.

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{5-1}\int_1^5 \operatorname{arcsec} x\,dx$$

$$\approx \frac{1}{4}\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x\right]$$

$$= \frac{1}{\cancel{4}\,2}\left[(\operatorname{arcsec} 2)\cancel{2} + (\operatorname{arcsec} 4)\cancel{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \operatorname{arcsec} 4\right]$$

Question 5. *(5 marks)* Evaluate the expression and simplify:

$$\frac{d}{dx}\left[\underbrace{\int_{\pi+x}^{\cot 3x}(\pi - u)(\operatorname{arccot} u)^7\,du}_{h(x)}\right]$$

$$h(x) = \int_{\pi+x}^{0}(\pi-u)(\operatorname{arccot} u)^7\,du + \int_0^{\cot 3x}(\pi-u)(\operatorname{arccot} u)^7\,du$$

$$= -\int_0^{\pi+x}(\pi-u)(\operatorname{arccot} u)^7\,du + \int_0^{\cot 3x}(\pi-u)(\operatorname{arccot} u)^7\,du$$

$$= -f(g_1(x)) + f(g_2(x)) \quad \text{where} \quad f(x) = \int_0^x (\pi-u)(\operatorname{arccot} u)^7\,du$$

$$g_1(x) = \pi + x$$

$$g_2(x) = \cot 3x$$

$$h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x) \quad \text{where} \quad f'(x) = (\pi - x)(\operatorname{arccot} x)^7$$

by 2^{nd} FTC

$$g_1'(x) = 1$$

$$g_2'(x) = -\csc^2 3x \cdot 3$$

$$= -\left[(\pi - (\pi + x))(\operatorname{arccot}(\pi + x))^7 \cdot 1\right] +$$

$$\left[(\pi - \cot 3x)(\operatorname{arccot}(\cot 3x))^7 \cdot -\csc^2 3x \cdot 3\right]$$

$$= x(\operatorname{arccot}(\pi+x))^7 - 3\csc^2 3x\,(\pi - \cot 3x)(3x)^7$$

Question 6. *(5 marks)* Evaluate the indefinite integral:

$\int x\ln(x^2+1)\,dx \longrightarrow = uv - \int v\,du$

$u = \ln(x^2+1)$ $\quad du = \frac{1}{x^2+1} \cdot 2x\,dx$

$v = \frac{x^2}{2}$ $\quad dv = x\,dx$

$$= \frac{x^2}{2}\ln(x^2+1) - \int \frac{x^2}{\cancel{2}} \frac{1}{x^2+1} \cancel{2}x\,dx$$

$$= \frac{x^2}{2}\ln(x^2+1) - \int \frac{x^3}{x^2+1}\,dx$$

$$\begin{array}{r} x \\ x^2+0x+1 \overline{)\, x^3+0x^2+0x+0} \\ -(x^3+0x^2+x) \\ \hline -x \end{array}$$

$$= \frac{x^2}{2}\ln(x^2+1) - \int x - \frac{x}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\ln(x^2+1) - \frac{x^2}{2} + \int \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \frac{x^2}{2}\ln(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{x^2}{2}\ln(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\ln|u| + C$$

$$= \frac{x^2}{2}\ln(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\ln(x^2+1) + C$$

Question 7. *(5 marks)* Suppose that $f(1) = 1, f(2) = 2, f'(1) = 3, f'(2) = 4$ and f'' is continuous. Find the value of $\int_1^2 xf''(x)\,dx$.

$$\int_1^2 x f''(x)\,dx = \Big[uv\Big]_1^2 - \int_1^2 v\,du$$

$u = x$ $\quad du = dx$

$v = f'(x)$ $\quad dv = f''(x)\,dx$

$$= \Big[x f'(x)\Big]_1^2 - \int_1^2 f'(x)\,dx$$

$$= \Big[2f'(2) - 1f'(1)\Big] - \Big[f(x)\Big]_1^2$$

$$= \Big[2\cdot 4 - 1\cdot 3\Big] - \Big[f(2) - f(1)\Big]$$

$$= [8-3] - [2-1]$$

$$= 5 - 1$$

$$= 4$$

Question 8. *(5 marks)* Prove: If $f(x)$ is an even integrable function on $[-a, a]$ then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(-x)dx + \int_{0}^{a} f(x)dx \quad \text{since } f(x) \text{ is even}$$

$u = -x$
$du = -dx$
$-du = dx$
$u(0) = 0$
$u(-a) = -(-a)$
$= a$

$$= -\int_{0}^{a} f(u)(-du) + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \quad \blacksquare$$

Question 9. *(3 marks)* Prove: If $f(x)$ is a continous function and c is a constant then

$$\int_{a}^{b} cf(x)\,dx = c\int_{a}^{b} f(x)\,dx$$

$$\int_{a}^{b} cf(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i^*)\,\Delta x_i$$

$$= \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$= c\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$= c\int_{a}^{b} f(x)dx$$

$\blacksquare$

Bonus Question.

Prove: If f is a continuous on [a,b], then the function g defined by

$$g(x) = \int_a^x f(t)\, dt \quad a \leq x \leq b$$

is an antiderivative of f, that is, g'(x)=f(x) for $a < x < b$

a. *(1 mark)* Use the limit definition of the derivative,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

and simplify the numerator of quotient using integral properties.

b. *(1 mark)* Suppose $h > 0$ and since f is continous on $[x, x+h]$ there exists values x_m and x_M such that $f(x_m)$ and $f(x_M)$ are the minimum and maximum of the function, respectively on $[x, x+h]$. Use $f(x_m)$ and $f(x_M)$ to bound above and below the quotient of the limit of part a.

c. *(1 mark)* Assume the inequality is also true for $h < 0$. Apply the Squeeze Theorem to complete the proof.

Thm: Second Fundamental Theorem of Calculus

If $f(t)$ is continuous on an interval I containing 'a', then

$$g'(x) = f(x)$$

proof:

recall: $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

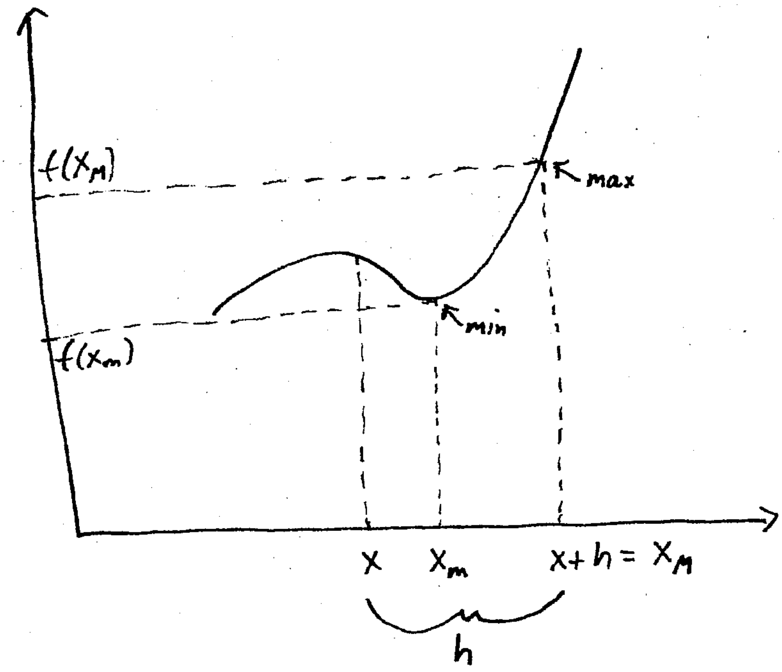
So using the above definition, we will show $g'(x) = f(x)$.

First we notice

(a.)
$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$
$$= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt$$
$$= \int_x^{x+h} f(t)dt$$

We will now use the squeeze thm. in order to show $g'(x) = f(x)$. So we need to bound $\int_x^{x+h} f(t)dt$ above and below.

(b.)



$$\therefore \quad f(x_m)h \le \int_x^{x+h} f(t)\,dt \le f(x_M)h$$

$$f(x_m) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x_M)$$

then by the squeeze theorem.

(c.)

$$\lim_{h\to 0} f(x_m) \le \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \lim_{h\to 0} f(x_M)$$

$$\lim_{h\to 0} f(x_m) \le g'(x) \le \lim_{h\to 0} f(x_M)$$

$$f(x) \le g'(x) \le f(x)$$

$$\therefore \ g'(x) = f(x)$$

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