Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S4: Winter 2011

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Test 2

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(3 marks)* Evaluate the definite integral:

$$\int_{\pi/12}^{\pi/9} \sin^2 3\alpha \, d\alpha. = \int_{\pi/12}^{\pi/9} \frac{1-\cos 6\alpha}{2} \, d\alpha$$

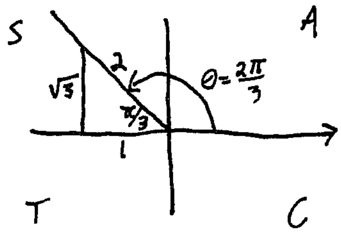
$$= \left[\frac{1}{2}\alpha - \frac{\sin 6\alpha}{12} \right]_{\pi/12}^{\pi/9}$$

$$= \left[\frac{1}{2}\cdot\frac{\pi}{9} - \frac{\sin\left(6\cdot\frac{\pi}{9}\right)}{12} \right] - \left[\frac{1}{2}\cdot\frac{\pi}{12} - \frac{\sin\left(6\cdot\frac{\pi}{12}\right)}{12} \right]$$

$$= \left[\frac{\pi}{18} - \frac{\sin\frac{2\pi}{3}}{12} \right] - \left[\frac{\pi}{24} - \frac{\sin\frac{\pi}{2}}{12} \right]$$

$$= \left[\frac{\pi}{18} - \frac{\frac{\sqrt{3}}{2}}{12} \right] - \left[\frac{\pi}{24} - \frac{1}{12} \right]$$

$$= \frac{\pi}{72} + \frac{2-\sqrt{3}}{24}$$



Question 2. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_{-\infty}^{\infty} \frac{1}{2x^2-4x+4} \, dx$$

$$= \int_{-\infty}^{0} \frac{1}{2x^2-4x+4} \, dx + \int_{0}^{\infty} \frac{1}{2x^2-4x+4} \, dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{2x^2-4x+4} \, dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{2x^2-4x+4} \, dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{2[(x-1)^2+1]} \, dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{2[(x-1)^2+1]} \, dx$$

$$= \frac{1}{2}\left[\lim_{a\to-\infty} \int_{a-1}^{-1} \frac{1}{u^2+1} \, du + \lim_{b\to\infty} \int_{-1}^{b-1} \frac{1}{u^2+1} \, du \right]$$

$$= \frac{1}{2}\left[\lim_{a\to-\infty} \left[\arctan u \right]_{a-1}^{-1} + \lim_{b\to\infty} \left[\arctan u \right]_{-1}^{b-1} \right]$$

$$= \frac{1}{2}\left[\lim_{a\to-\infty} \left[\arctan(-1) - \overset{-\frac{\pi}{2}}{\arctan(a-1)} \right] + \lim_{b\to\infty} \left[\overset{\frac{\pi}{2}}{\arctan(b-1)} - \arctan(-1) \right] \right]$$

$$= \frac{1}{2}\left[\pi\right] = \frac{\pi}{2}$$

$$2x^2-4x+4$$
$$= 2\left[x^2-2x+2\right]$$
$$= 2\left[(x^2-2x+1)-1+2\right]$$
$$= 2\left[(x-1)^2+1\right]$$

$u = x-1$
$du = dx$
$u(0) = -1$
$u(a) = a-1$
$u(b) = b-1$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int \csc^3 5\theta \cot^3 5\theta \, d\theta = \int \csc^2 5\theta \cot^2 5\theta \csc 5\theta \cot 5\theta \, d\theta$$

$$= \int \csc^2 5\theta (\csc^2 5\theta - 1) \csc 5\theta \cot 5\theta \, d\theta$$

$u = \csc 5\theta$

$du = -\csc 5\theta \cot 5\theta \cdot 5 \, d\theta$

$\frac{-du}{5} = \csc 5\theta \cot 5\theta \, d\theta$

$$= \int u^2(u^2-1) \frac{-du}{5}$$

$$= \frac{-1}{5}\int u^4 - u^2 \, du$$

$$= \frac{-1}{5}\left[\frac{u^5}{5} - \frac{u^3}{3}\right] + C$$

$$= \frac{-1}{5}\left[\frac{\csc^5 5\theta}{5} - \frac{\csc^3 5\theta}{3}\right] + C$$

$$= \frac{\csc^3 5\theta}{15} - \frac{\csc^5 5\theta}{25} + C$$

Question 4. *(5 marks)* Find the length of the curve.

$y = \ln(\sec x)$, $0 \le x \le \frac{\pi}{4}$

$$y' = \frac{1}{\cancel{\sec x}} \cancel{\sec x} \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1 + (y')^2} \, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx$$

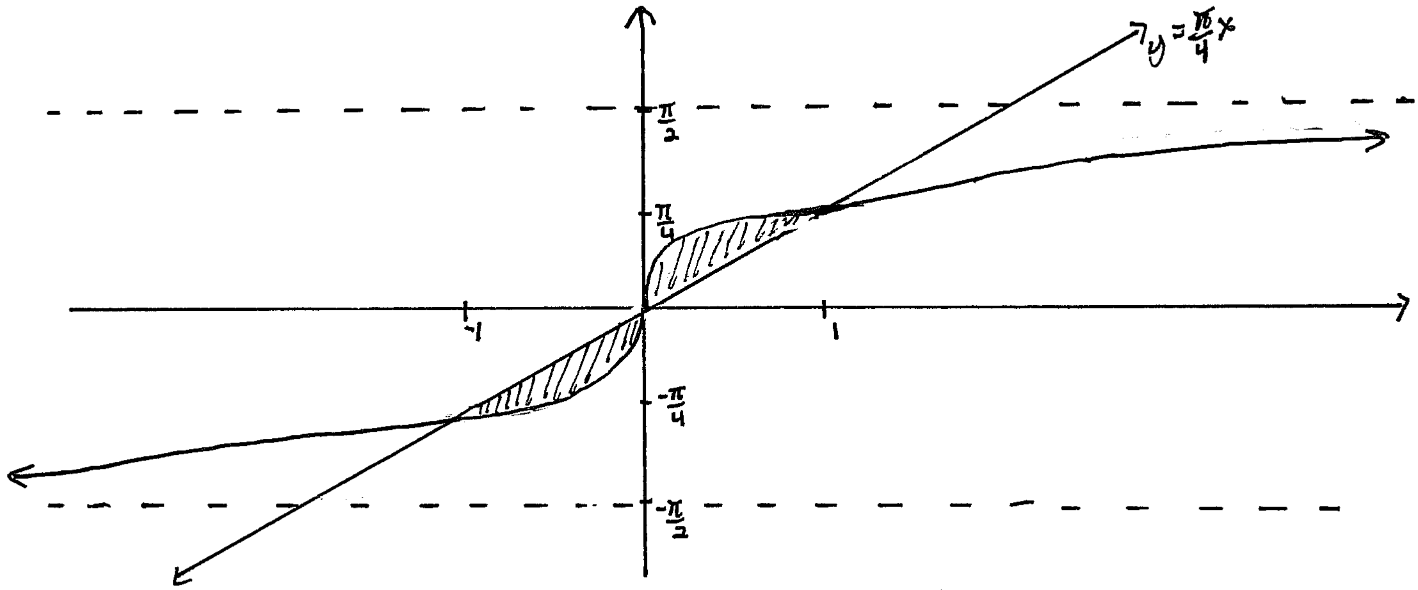
$$= \int_0^{\pi/4} |\sec x| \, dx$$

$$= \int_0^{\pi/4} \sec x \, dx \quad \text{since } \sec x > 0 \ \forall x \in [0, \tfrac{\pi}{4}]$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} = \ln|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln|1| = \ln|\sqrt{2}+1|$$

Question 5. *(5 marks)* Sketch and find the total area of the region(s) bounded by the graphs of $y = \arctan x$, $y = \frac{\pi}{4}x$.



$$A = \int_{-1}^{0} \frac{\pi}{4}x - \arctan x \, dx + \int_{0}^{1} \arctan x - \frac{\pi}{4}x \, dx$$

$$= \int_{-1}^{0} \frac{\pi}{4}x \, dx - \int_{-1}^{0} \arctan x \, dx + \int_{0}^{1} \arctan x \, dx - \int_{0}^{1} \frac{\pi}{4}x \, dx$$

$$= \left[\frac{\pi x^2}{8}\right]_{-1}^{0} + \int_{0}^{1} \arctan x \, dx - \int_{-1}^{0} \arctan x \, dx - \left[\frac{\pi x^2}{8}\right]_{0}^{1}$$

$$= -\frac{\pi}{4} + \left[uv\right]_{0}^{1} - \int_{0}^{1} v \, du - \left[\left[uv\right]_{-1}^{0} - \int_{-1}^{0} v \, du\right]$$

$u = \arctan x \qquad du = \frac{1}{x^2+1} dx$

$v = x \qquad dv = dx$

$$= \frac{-\pi}{4} + \left[x \arctan x\right]_{0}^{1} - \int_{0}^{1} \frac{x}{x^2+1} dx - \left[x \arctan x\right]_{-1}^{0} + \int_{-1}^{0} \frac{x}{x^2+1} dx$$

$$= \frac{-\pi}{4} + \arctan 1 - 0 \arctan 0 - \left[\frac{1}{2}\ln(x^2+1)\right]_{0}^{1} - 0 \arctan 0 + (-1)\arctan(-1) + \left[\frac{1}{2}\ln(x^2+1)\right]_{-1}^{0}$$

$$= \frac{-\pi}{4} + \frac{\pi}{4} - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1 + \frac{\pi}{4} + \frac{1}{2}\ln 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} - \ln 2$$

or integrate with respect to y.

$$y = \frac{\pi}{4} x$$

$$\frac{4}{\pi} y = x$$

$$y = \arctan x$$

$$\tan y = \tan \arctan x$$

$$\tan y = x$$

$$A = \int_{-\frac{\pi}{4}}^{0} \tan y - \frac{4}{\pi} y \, dy$$

$$+ \int_{0}^{\pi/4} \frac{4}{\pi} y - \tan y \, dy$$

$$= \left[-\ln|\cos y| - \frac{4}{\pi}\frac{y^2}{2} \right]_{-\frac{\pi}{4}}^{0} + \left[\frac{4}{\pi}\frac{y^2}{2} + \ln|\cos y| \right]_{0}^{\pi/4}$$

$$= \left[-\overbrace{\ln|\cos 0|}^{0} - \underbrace{\frac{4}{\pi}\frac{0^2}{2}}_{0} \right] - \left[-\ln\left|\cos \frac{-\pi}{4}\right| - \frac{4}{\pi}\frac{\left(\frac{-\pi}{4}\right)^2}{2} \right]$$

$$+ \left[\frac{4}{\pi}\frac{\left(\frac{\pi}{4}\right)^2}{2} + \ln\left|\cos\frac{\pi}{4}\right| \right] - \left[\overbrace{\frac{4}{\pi}\frac{0^2}{2}}^{0} + \overbrace{\ln|\cos 0|}^{0} \right]$$

$$= \frac{\pi}{8} + \frac{\pi}{8} + \ln\frac{1}{\sqrt{2}} + \ln\frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{4} - \ln 2$$

Question 7. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{3x^2+3x+2}{x^3+2x}\,dx$$

$$\frac{3x^2+3x+2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$\frac{(3x^2+3x+2)\cancel{(x)}\cancel{(x^2+2)}}{\cancel{x}\cancel{(x^2+2)}} = \frac{A\cancel{x}(x^2+2)}{\cancel{x}} + \frac{(Bx+C)(x)\cancel{(x^2+2)}}{\cancel{x^2+2}}$$

$$3x^2+3x+2 = A(x^2+2) + (Bx+C)x$$

Let $x=0$

$$3 0^2 + 3\cdot 0 + 2 = A(0^2+2) + (B(0)+C)0$$

$$2 = 2A$$

$$1 = A$$

Let $x=1$

$$3 1^2 + 3\cdot 1 + 2 = A(1^2+2) + (B(1)+C)(1)$$

$$8 = 1(3) + B + C$$

$$5 = B + C \quad (1)$$

Let $x=-1$

$$3\cdot(-1)^2 + 3(-1) + 2 = A((-1)^2+2) + (B(-1)+C)(-1)$$

$$2 = 1(3) + B - C$$

$$-1 = B - C$$

$$-1 + C = B \quad (2)$$

Sub (2) in (1)

$$5 = -1 + C + C$$

$$6 = 2C$$

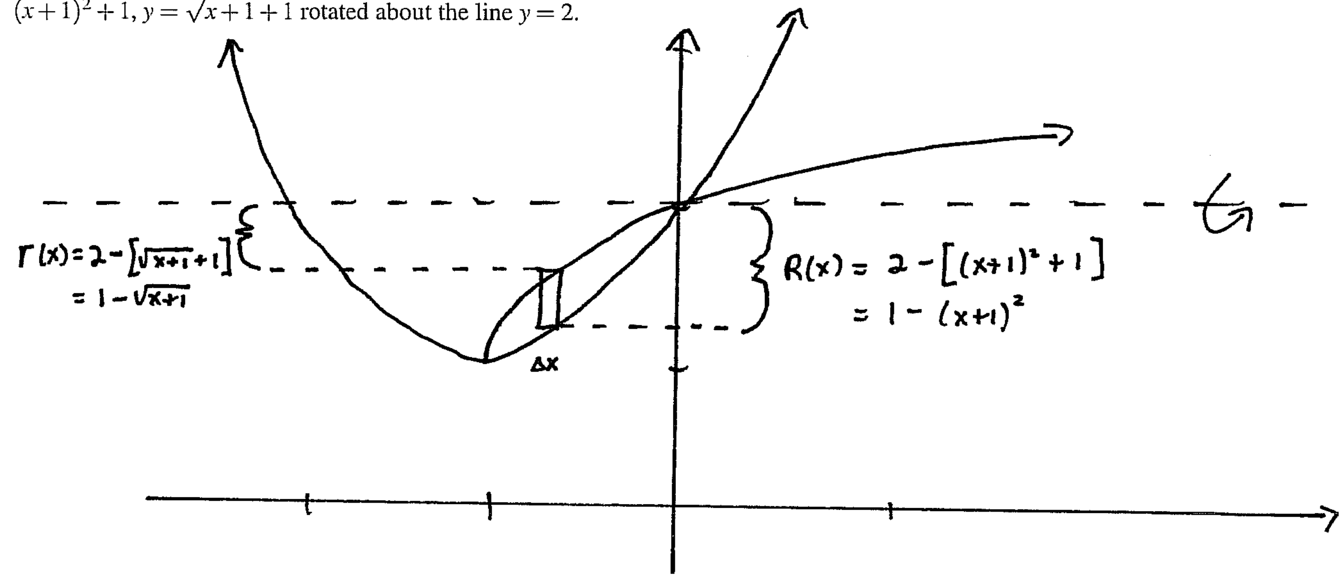
$$3 = C$$

$\therefore \quad B = 2$

So $\displaystyle \int \frac{1}{x} + \frac{2x+3}{x^2+2}\,dx = \int \frac{1}{x}\,dx + \int \frac{2x}{x^2+2}\,dx + \int \frac{3}{x^2+(\sqrt{2})^2}\,dx$

$$= \ln|x| + \ln(x^2+2) + \frac{3}{\sqrt{2}} \arctan\frac{x}{\sqrt{2}} + C$$

Question 8. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = (x+1)^2+1$, $y = \sqrt{x+1}+1$ rotated about the line $y = 2$.



intersection of two curves:

$$(x+1)^2 + 1 = \sqrt{x+1} + 1$$
$$(x+1)^2 = \sqrt{x+1}$$
$$(x+1)^4 = x+1$$
$$0 = (x+1)^4 - (x+1)$$
$$0 = (x+1)\left[(x+1)^3 - 1\right]$$
$$0 = (x+1)\left[x^3 + 3x^2 + 3x + 1 - 1\right]$$
$$0 = (x+1)\left[x\,(x^2+3x+3)\right]$$

$x=-1$ $\quad$ $x=0$ $\quad$ irreducible

<u>rep. element:</u>

$$\Delta V = \pi\left[(R(x))^2 - (r(x))^2\right]\Delta x$$
$$= \pi\left[\left(1-(x+1)^2\right)^2 - \left(1-\sqrt{x+1}\right)^2\right]\Delta x$$

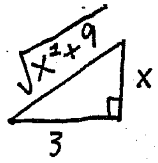
$$V = \int_{-1}^{0} \pi\left[\left(1-(x+1)^2\right)^2 - \left(1-\sqrt{x+1}\right)^2\right] dx$$

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{81\sqrt{9x^2+81}}{x^6}\,dx = \int \frac{81\sqrt{9(x^2+9)}}{x^6}\,dx$$

$x = 3\tan\theta \longrightarrow \frac{x}{3} = \tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$

$$= \int \frac{243\sqrt{x^2+9}}{x^6}\,dx$$

$$= \int \frac{243\sqrt{(3\tan\theta)^2+9}}{(3\tan\theta)^6}\,3\sec^2\theta\, d\theta$$

$$= \int \frac{\cancel{3^6}\sqrt{9(\tan^2\theta+1)}}{\cancel{3^6}\tan^6\theta}\sec^2\theta\, d\theta$$

$$= \int \frac{\sqrt{9\sec^2\theta}}{\tan^6\theta}\sec^2\theta\, d\theta$$

$$= \int \frac{3\sec^3\theta}{\tan^6\theta}\,d\theta$$

$$= 3\int \frac{1}{\cancel{\cos^3\theta}}\,\frac{\cancel{\cos^6}^{\,3}\theta}{\sin^6\theta}\,d\theta$$

$$= 3\int \frac{\cos^2\theta}{\sin^6\theta}\cos\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= 3\int \frac{1-\sin^2\theta}{\sin^6\theta}\cos\theta\, d\theta$$

$$= 3\int \frac{1-u^2}{u^6}\,du$$

$$\rightarrow 3\int \frac{1}{u^6} - \frac{1}{u^4}\,du$$

$$= 3\left[\frac{-1}{5u^5} + \frac{1}{3u^3}\right] + C$$

$$= 3\left[\frac{-1}{5\sin^5\theta} + \frac{1}{3\sin^3\theta}\right] + C$$

$$= 3\left[\frac{-(\sqrt{x^2+9})^5}{5x^5} + \frac{(\sqrt{x^2+9})^3}{3x^3}\right] + C$$

$$= \frac{-3(\sqrt{x^2+9})^5}{5x^5} + \frac{(\sqrt{x^2+9})^3}{x^3} + C$$

Question 9. *(5 marks)* For what values of m do the line $y = mx$ and the curve

$$y = \frac{x}{x^2+1}$$

enclose a region? Find the area of the region.

Lets find the intersection between the two curves

$$mx = \frac{x}{x^2+1}$$

$$x = mx(x^2+1)$$

$$x = mx^3 + mx$$

$$0 = mx^3 + mx - x$$

$$0 = x(mx^2 + m - 1)$$

$$x = 0 \qquad\qquad mx^2 + m - 1 = 0$$

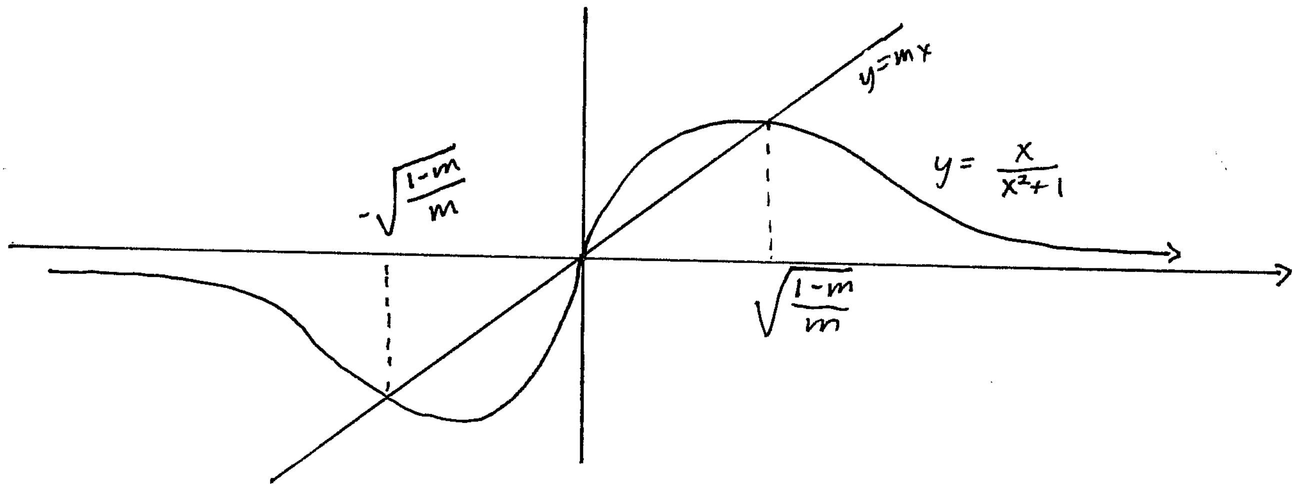
$$mx^2 = 1-m$$

$$x^2 = \frac{1-m}{m}$$

So $\frac{1-m}{m} > 0$ for two other intersection to exists

$\therefore$ $0 < m < 1$

and $x = \pm\sqrt{\frac{1-m}{m}}$



$$A = \int_{-\sqrt{\frac{1-m}{m}}}^{0} mx - \frac{x}{x^2+1}\,dx + \int_0^{\sqrt{\frac{1-m}{m}}} \frac{x}{x^2+1} - mx\,dx$$

$$= \left[\frac{mx^2}{2} - \frac{1}{2}\ln(x^2+1)\right]_{-\sqrt{\frac{1-m}{m}}}^{0} + \left[\frac{1}{2}\ln(x^2+1) - \frac{mx^2}{2}\right]_0^{\sqrt{\frac{1-m}{m}}}$$

$$= \frac{-m\left(-\sqrt{\frac{1-m}{m}}\right)^2}{2} + \frac{1}{2}\ln\left(\left(-\sqrt{\frac{1-m}{m}}\right)^2+1\right) + \frac{1}{2}\ln\left(\left(\sqrt{\frac{1-m}{m}}\right)^2+1\right) - \frac{m\left(\sqrt{\frac{1-m}{m}}\right)^2}{2}$$

$$= \frac{1-m}{2} + \frac{1}{2}\ln\left(\frac{1}{m}\right) + \frac{1}{2}\ln\left(\frac{1}{m}\right) + \frac{1-m}{2} = 1-m+\ln\left(\frac{1}{m}\right)$$

Bonus Question. *(5 marks)* Evaluate the limit.

$$\lim_{h\to 0} \frac{\int_1^{\cos h} \arctan x\, dx}{3e^{-h} - \ln(h+1) + \sin(4h) - 3} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\frac{d}{dh}\left[\int_1^{\cos h} \arctan x\, dx\right]}{\frac{d}{dh}\left[3e^{-h} - \ln(h+1) + \sin(4h) - 3\right]} \quad \text{by } \hat{H}$$

$$f(g(\ )) = \int_1^{\cos h} \arctan x\, dx$$

$$= \lim_{h\to 0} \frac{\arctan(\cos h)(-\sin h)}{-3e^{-h} - \frac{1}{h+1} + \cos(4h)} \quad \text{I.F. } \frac{0}{0}$$

where $f(h) = \int_1^{h} \arctan x\, dx$

$g(h) = \cos h$

$$= \lim_{h\to 0} \frac{\frac{d}{dh}\left[-\arctan(\cos h)\sin h\right]}{\frac{d}{dh}\left[-3e^{-h} - \frac{1}{h+1} + 4\cos 4h\right]} \quad \text{by } \hat{H}$$

So $\frac{d}{dh}\left[f(g(h))\right]$

$= f'(g(\ ))g'(h)$

$$= \lim_{h\to 0} \frac{\frac{-1}{1+\cos^2 h}(-\sin h)\sin h - \arctan(\cos h)\cos h}{3e^{-h} + \frac{1}{(h+1)^2} - 16\sin 4h}$$

where $f'(h) = \arctan h$ by 2^{nd} FTC and $g'(h) = -\sin h$

$$= \frac{\frac{-\pi}{4}}{4}$$

$$= \frac{-\pi}{16}$$