Dawson College: Calculus II (SCIENCE): 201-NYB-05-S4: Winter 2011

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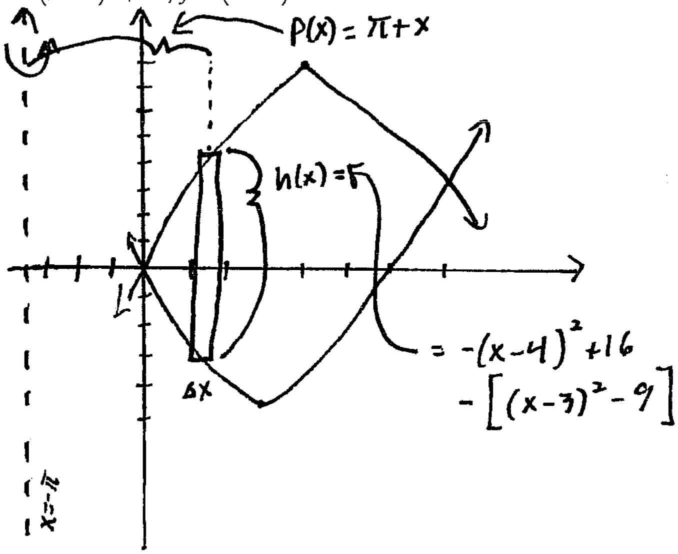
Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = -(x-4)^2+16$, $y=(x-3)^2-9$ rotated about the line $x=-\pi$.



Lets find the intersection of the two curves

$$-(x-4)^2+16 = (x-3)^2-9$$
$$-x^2+8x = x^2-6x$$
$$0 = 2x^2-14x$$
$$0 = x^2-7x$$
$$0 = x(x-7)$$
$$x=0 \qquad x=7$$

rep. element:

$$\Delta V = 2\pi p(x)\, h(x)\, \Delta x$$
$$= 2\pi(\pi+x)\left[-(x-4)^2+16-\left[(x-3)^2-9\right]\right]\Delta x$$

$$V = \int_0^7 2\pi(\pi+x)\left[-(x-4)+16-\left[(x-3)^2-9\right]\right]dx$$

Question 2. *(5 marks)* Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \left[\sec\left(\frac{1}{n}\right)\right]^n$$

$$\lim_{n \to \infty} \left(\sec\left(\frac{1}{n}\right)\right)^n \qquad \text{I.F. } 1^\infty$$

$$y = \lim_{x \to \infty} \left(\sec\left(\frac{1}{x}\right)\right)^x$$

$$\ln y = \ln \lim_{x \to \infty} \left(\sec\left(\frac{1}{x}\right)\right)^x$$

$$\ln y = \lim_{x \to \infty} \ln \left(\sec\left(\frac{1}{x}\right)\right)^x$$

$$\ln y = \lim_{x \to \infty} x \ln \left(\sec\left(\frac{1}{x}\right)\right) \qquad \text{I.F. } \infty \cdot 0$$

$$\ln y = \lim_{x \to \infty} \frac{\ln\left(\sec\left(\frac{1}{x}\right)\right)}{\frac{1}{x}} \qquad \text{I.F. } \frac{0}{0}$$

$$\ln y = \lim_{x \to \infty} \frac{\frac{1}{\cancel{\sec\left(\frac{1}{x}\right)}} \cancel{\sec\left(\frac{1}{x}\right)} \tan\left(\frac{1}{x}\right) \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} \qquad \text{by } \hat{H}$$

$$\ln y = \tan(0)$$

$$\ln y = 0$$

$$y = 1$$

$\therefore \quad a_n \to 1 \qquad \text{as } n \to \infty.$

Question 3. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1+3^{n-1}}{\pi^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\pi^{n+1}} + \sum_{n=1}^{\infty} \frac{3^{n-1}}{\pi^{n+1}}$$

$$= \frac{1}{\pi(\pi-1)} + \frac{1}{\pi(\pi-3)} = \frac{2\pi-4}{\pi(\pi-1)(\pi-3)}$$

$$\sum_{n=0}^{\infty} \frac{1}{\pi\pi^n} - a_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{\pi}\left(\frac{1}{\pi}\right)^n - \frac{1}{\pi\pi^0}$$

$$= \frac{1/\pi}{1-1/\pi} - 1/\pi$$ Converges since $r = \frac{1}{\pi} < 1$

$$= \frac{1/\pi}{\frac{\pi-1}{\pi}} - 1/\pi$$

$$= \frac{1}{\pi-1} - \frac{1}{\pi} = \frac{1}{\pi(\pi-1)}$$

$$\sum_{n=0}^{\infty} \frac{3^n 3^{-1}}{\pi^n \pi} - b_0$$

$$= \sum_{n=0}^{\infty} \frac{1}{3\pi}\left(\frac{3}{\pi}\right)^n - \frac{1}{3\pi}\left(\frac{3}{\pi}\right)^0$$

$$= \frac{1/3\pi}{1-3/\pi} - \frac{1}{3\pi}$$ converges since $r = \frac{3}{\pi} < 1$

$$= \frac{1}{3(\pi-3)} - \frac{1}{3\pi}$$

$$= \frac{9}{9\pi(\pi-3)} = \frac{1}{\pi(\pi-3)}$$

Question 4. *(5 marks)* Determine whether the series is absolutely convergent, or conditionally convergent, or divergent.

$$\sum_{n=5}^{\infty} \frac{n5^n(-1)^n}{n!}$$

Lets determine if it converges absolutely by using the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)5^{n+1}(-1)^{n+1}}{(n+1)!}}{\frac{n5^n(-1)^n}{n!}}\right| = \lim_{n\to\infty} \frac{(n+1)5^{n+1}}{(n+1)!}\cdot\frac{n!}{n5^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}}{n}\cdot\frac{\cancel{n!}}{\cancel{(n+1)}\cancel{(n!)}}\cdot\frac{\cancel{5^n}5}{\cancel{5^n}}$$

$$= \lim_{n\to\infty} \frac{5}{n}$$

$$= 0 < 1$$

$\therefore$ converges absolutely by ratio test.

Question 5. *(5 marks)* Determine whether the series is absolutely convergent, or conditionally convergent, or divergent.

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$$

Lets determine if the series is absolutely convergent

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[5]{n}} \right| = \sum_{n=3}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=3}^{\infty} \frac{1}{n^{1/5}}$$

diverges since p-series where $p = 1/5 < 1$.

$\therefore$ not absolutely convergent

Lets apply the alternating series test to determine if the series is conditionally convergent

Let $\sum_{n=3}^{\infty} (-1)^{n+1} b_n$ where $b_n = \frac{1}{\sqrt[5]{n}}$.

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[5]{n}} = 0$

(ii) $b_{n+1} \overset{?}{<} b_n$, let $f(x) = \frac{1}{\sqrt[5]{x}}$, notice $f'(x) = \frac{-1/5}{\sqrt[5]{x^6}} < 0$

for $x > 0$

$\therefore f(n+1) < f(n)$

$b_{n+1} < b_n$

$\therefore$ convergent by alternating series test

$\therefore$ conditionally convergent.

Question 6. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n^3+n^4+10}}{\sqrt[3]{n^9+3n^4+2}}$$

Let $a_n = \dfrac{\sqrt{n^3+n^4+10}}{\sqrt[3]{n^9+3n^4+2}}$.

Let $\sum_{n=2}^{\infty} b_n$ where $b_n = \dfrac{\sqrt{n^4}}{\sqrt[3]{n^9}} = \dfrac{n^2}{n^3} = \dfrac{1}{n}$, diverges

since p-series where $p=1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^3+n^4+10}}{\sqrt[3]{n^9+3n^4+2}}}{\frac{\sqrt{n^3}}{\sqrt[3]{n^9}}} = \lim_{n\to\infty} \frac{\sqrt{n^3+n^4+10}}{\sqrt[3]{n^9+3n^4+2}} \cdot \frac{\sqrt[3]{n^9}}{\sqrt{n^4}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^9}}{\sqrt[3]{n^9+3n^4+2}} \cdot \frac{\sqrt{n^3+n^4+10}}{\sqrt{n^4}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^9}{n^9+3n^4+2}} \cdot \lim_{n\to\infty} \sqrt{\frac{n^3+n^4+10}{n^4}}$$

$= 1 > 0$ and finite

$\therefore \sum_{n=2}^{\infty} a_n$ diverges by limit comparison test since

$\sum_{n=2}^{\infty} b_n$ diverges.

Question 7. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\operatorname{arcsec}(n)}{n\sqrt{n^2-1}}$$

note: $\lim_{n\to\infty} \operatorname{arcsec} n = \frac{\pi}{2}$ and increasing since $f'(x) = \frac{1}{x\sqrt{x^2-1}} > 0$

Solution #1:

Let $a_n = \frac{\operatorname{arcsec} n}{n\sqrt{n^2-1}}$

So $a_n = \frac{\operatorname{arcsec} n}{n\sqrt{n^2-1}} \le \frac{\pi/2}{n\sqrt{n^2-1}} \le \frac{\pi/2}{n\sqrt{n^2-\frac{1}{4}n^2}} = \frac{\pi}{2n\sqrt{\frac{3}{4}n^2}}$

$= \frac{\pi}{\sqrt{3}} \frac{1}{n^2} = b_n$

note: $\sum_{n=2}^{\infty} b_n$ converges since p-series where $p = 2 > 1$

$\therefore$ by comparison test $\sum_{n=2}^{\infty} a_n$ converges

Solution #2:

Let $a_n = \frac{\operatorname{arcsec} n}{n\sqrt{n^2-1}}$

So $a_n = \frac{\operatorname{arcsec} n}{n\sqrt{n^2-1}} \le \frac{\pi/2}{n\sqrt{n^2-1}} = b_n$

Let $c_n = \frac{1}{n^2} = \frac{1}{n\sqrt{n^2}}$ then $\sum_{n=2}^{\infty} c_n$ converges since p-series

where $p = 2 > 1$ and

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{\frac{\pi/2}{n\sqrt{n^2-1}}}{\frac{1}{n\sqrt{n^2}}} = \lim_{n\to\infty} \frac{\pi/2}{\cancel{n}\sqrt{n^2-1}} \cdot \frac{\cancel{n}\sqrt{n^2}}{1}$$

$$= \frac{\pi}{2} \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}}$$

$= \frac{\pi}{2} > 0$ and finite

$\therefore$ by limit comparison test $\sum_{n=2}^{\infty} b_n$ converges since

$\sum_{n=2}^{\infty} c_n$ converges

$\therefore \sum_{n=2}^{\infty} a_n$ converges by comparison test.

Solution #3

Let $f(x) = \dfrac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}$

- $f(x)$ is continuous for $x \geq 2$.
- $f(x)$ is positive for $x \geq 2$
- $f'(x) = \dfrac{1}{(x\sqrt{x^2-1})^2}\left[\dfrac{1}{\cancel{x\sqrt{x^2-1}}} \cdot \cancel{x\sqrt{x^2-1}} - \operatorname{arcsec} x\left[\sqrt{x^2-1} + x\,\dfrac{1}{\cancel{2}\sqrt{x^2-1}}\,\cancel{2}x\right]\right]$

$$= \frac{1}{(x\sqrt{x^2-1})^2}\left[1 - \operatorname{arcsec} x\left[\frac{x^2-1+x^2}{\sqrt{x^2-1}}\right]\right] < 0$$

since $\operatorname{arcsec} x$ is increasing and $\operatorname{arcsec} 2 = \frac{\pi}{3}$ for $x \geq 2$

$$\int_2^{\infty} \frac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}\,dx = \lim_{b\to\infty}\int_2^b \frac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}\,dx$$

$u = \operatorname{arcsec} x$

$du = \dfrac{1}{x\sqrt{x^2-1}}\,dx$

$u(2) = \operatorname{arcsec} 2 = \pi/3$

$u(b) = \operatorname{arcsec} b$

$$= \lim_{b\to\infty}\int_{\frac{\pi}{3}}^{\operatorname{arcsec} b} u\,du$$

$$= \lim_{b\to\infty}\left[\frac{u^2}{2}\right]_{\pi/3}^{\operatorname{arcsec} b} = \lim_{b\to\infty}\left[\frac{(\operatorname{arcsec} b)^2}{2} - \frac{(\pi/3)^2}{2}\right]$$

$$= \frac{(\pi/2)^2}{2} - \frac{(\pi/3)^2}{2}$$

$\therefore \sum_{n=2}^{\infty} \dfrac{\operatorname{arcsec} n}{n\sqrt{n^2-1}}$ converges by integral test since the improper integral converges.

Question 8. *(5 marks)* Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$$

Let $a_n = \dfrac{(-1)^n(x+2)^n}{n2^n}$

Lets determine the radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n(x+2)^n}{n2^n}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\,\cancel{(x+2)^{n+1}}}{(n+1)\cancel{2^{n+1}}}\cdot\frac{n\cancel{2^n}}{\cancel{(-1)^n}\,\cancel{(x+2)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+2)\cdot n}{(n+1)\,2}\right|$$

$$= |x+2|\lim_{n\to\infty}\left|\frac{n}{2n+2}\right|$$

$$= \frac{1}{2}|x+2| < 1 \quad \text{for convergence}$$

$|x+2| < 2 = R$

$-2 < x+2 < 2$ $\quad\therefore$ the radius of convergence is 2.

$-4 < x < 0$

Lets verify the endpoint of the interval $(-4,0)$ for convergence

Let $x=-4$

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-4+2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n}\left(\frac{2}{2}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}$$

diverges since p-series where $p=1$

Let $x=0$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cancel{(0+2)^n}}{n\cancel{2^n}}$$

$$= \sum_{n=1}^{\infty}(-1)^n b_n$$

where $b_n = \dfrac{1}{n}$

- $\lim_{n\to\infty} b_n = 0$
- $b_{n+1} \overset{?}{\le} b_n$

$\dfrac{1}{n+1} \overset{?}{\le} \dfrac{1}{n}$

$n \le n+1$

$\therefore$ converges by alternating series test.

$\therefore$ interval of convergence is $(-4,0]$.

Question 9. *(5 marks)* Find the Taylor series for $f(x) = \frac{1}{x^2}$ centered at $x = 2$. Assume that f has a power series expansion. Do not show that $R_n \to 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$f^{(0)}(x) = \frac{1}{x^2} \qquad f^{(0)}(2) = \frac{1}{2^2}$$

$$f^{(1)}(x) = \frac{-2}{x^3} \qquad f^{(1)}(2) = \frac{-2}{3}$$

$$f^{(2)}(x) = \frac{-2(-3)}{x^4} \qquad f^{(2)}(2) = \frac{-2(-3)}{2^4}$$

$$f^{(3)}(x) = \frac{-2(-3)(-4)}{x^5} \qquad f^{(3)}(2) = \frac{-2(-3)(-4)}{2^5}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \qquad f^{(n)}(2) = \frac{(-1)^n (n+1)!}{2^{n+2}}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^{n+2}} \cdot \frac{(x-2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-2)^n}{2^{n+2}}$$

Bonus Question. *(5 marks)* Show that the series diverges.

$$\sum_{n=1}^{\infty} \frac{\int_1^{\cos\left(\frac{1}{n}\right)} \arctan x \, dx}{3e^{-1/n} - \ln\left(\frac{1}{n}+1\right) + \sin\left(\frac{4}{n}\right) - 3}$$

Let $f(t) = \dfrac{\int_1^{\cos(\frac{1}{t})} \arctan x \, dx}{3e^{-1/t} - \ln\left(\frac{1}{t}+1\right) + \sin\left(\frac{4}{t}\right) - 3}$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\int_1^{\cos(\frac{1}{t})} \arctan x \, dx}{3e^{-1/t} - \ln\left(\frac{1}{t}+1\right) + \sin\left(\frac{4}{t}\right) - 3} \quad \text{I.F. } \frac{0}{0}$$

by 2^{nd} FTC.

$$= \lim_{t\to\infty} \frac{\arctan\left[\cos\left(\frac{1}{t}\right)\right]\left(-\sin\left(\frac{1}{t}\right)\right)\frac{-1}{t^2}}{-3e^{-1/t}\cdot\frac{-1}{t^2} - \frac{1}{1/t+1}\cdot\frac{-1}{t^2} + \cos\left(\frac{4}{t}\right)4\left(\frac{-1}{t^2}\right)} \quad \text{by } \hat{H} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{t\to\infty} \frac{\frac{1}{1+\left[\cos\left(\frac{1}{t}\right)\right]^2}\cdot -\sin\left(\frac{1}{t}\right)\cdot\cancel{\frac{-1}{t^2}}\cdot -\sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)\left(\cancel{\frac{-1}{t^2}}\right)\arctan\left[\cos\left(\frac{1}{t}\right)\right]}{+3e^{-1/t}\left(\cancel{\frac{-1}{t^2}}\right) - \frac{-1}{(1/t+1)^2}\cdot\cancel{\frac{-1}{t^2}} - \sin\left(\frac{4}{t}\right)\cancel{\frac{-1}{t^2}}} \quad \text{by } \hat{H}$$

$$= \frac{-\cos(0)\arctan\left[\cos(0)\right]}{+3e^0 + \frac{1}{(0+1)^2}}$$

$$= \frac{-\pi/4}{4}$$

$$= -\pi/16 \neq 0$$

$\therefore$ by the n^{th} term divergence test the series diverges.