Dawson College: Calculus II - Commerce: 201-NYB-05 S13

February 23, 2012

**Last Name:** SOLUTIONS

**First Name:**

**Student ID:**

# Test 1

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

**Question 1.** Find/evaluate the following:

(a) *(3 marks)* $\int \frac{5-x^2-x}{x^2}\,dx = \int \left(5x^{-2} - 1 - \frac{1}{x}\right)dx$

$$= \frac{5x^{-1}}{-1} - x - \ln|x| + c$$

$$= -\frac{5}{x} - x - \ln|x| + c$$

(b) *(4 marks)* $\int x^2 e^{2x^3-1}\,dx$

LET $u = 2x^3 - 1$

$du = 6x^2\,dx$

$\frac{du}{6x^2} = dx$

$$= \int x^2 e^u \frac{du}{6x^2} = \frac{1}{6}\int e^u\,du$$

$$= \frac{1}{6}e^u + c$$

$$= \frac{1}{6}e^{2x^3-1} + c$$

(c) *(4 marks)* $\int \frac{1}{x(\ln x)}\,dx$

LET $u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$= \int \frac{1}{x\,u} \cdot x\,du = \int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln|\ln x| + c$

(d) *(4 marks)* $\int \frac{(x-1)^2}{x}\,dx = \int \frac{x^2 - 2x + 1}{x}\,dx$

$= \int \left(x - 2 + \frac{1}{x}\right) dx$

$= \frac{x^2}{2} - 2x + \ln|x| + c$

**Question 2.**

**(a)** *(1 marks)* Write the definition of the definite integral

$$\int_a^b f(x)\,dx \quad = \quad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

**(b)** *(3 marks)* Using words describe

**i)** What $\Delta x$ is.

$\Delta x$ IS THE WIDTH OF APPROXIMATING RECTANGLES IN THE RIEMANN SUM.

**ii)** What $f(x_i)$ is.

$f(x_i)$ IS THE HEIGHT THE APPROXIMATING RECTANGLE ON THE SUBINTERVAL $[x_{i-1}, x_i]$.

**iii)** What the purpose of the limit is.

THE SUM OF APPROXIMATING RECTANGLES $\sum_{i=1}^{n} f(x_i)\Delta x$ IS AN APPROXIMATION FOR THE "NET AREA" UNDER THE FUNCTION $f(x)$ ON $[a,b]$. THIS APPROXIMATION GETS BETTER AS $n$ GETS LARGER (MORE RECTANGLES). THE LIMIT IN THE DEFINITION GIVES US THE EXACT AREA.

**(c)** *(6 marks)* Use the definition of the definite integral to find:

$$\int_1^3 (1+2x-x^2)\,dx$$

- $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$ • $x_i = a + i\Delta x = 1 + \frac{2i}{n}$

- $f(x_i) = 1 + 2x_i - (x_i)^2 = 1 + 2\left(1+\frac{2i}{n}\right) - \left(1+\frac{2i}{n}\right)^2$

$= 1 + 2 + \frac{4i}{n} - \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) = 3 + \frac{4i}{n} - 1 - \frac{4i}{n} - \frac{4i^2}{n^2}$

$= 2 - \frac{4i^2}{n^2}$

- $f(x_i)\Delta x = \left(2 - \frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \frac{4}{n} - \frac{8i^2}{n^3}$

- $\sum_{i=1}^{n}\left(\frac{4}{n} - \frac{8i^2}{n^3}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{4}{n}\cdot n - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= 4 - \frac{4}{3}\cdot\frac{2n^2+3n+1}{n^2}$

$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[4 - \frac{4}{3}\cdot\frac{2 + 3/n + 1/n^2}{1}\right]$

$= 4 - \frac{4}{3}\cdot 2$

$= 4 - \frac{8}{3} =$

$= \frac{4}{3}$

$\therefore \int_1^3 (1+2x-x^2)\,dx = \frac{4}{3}$

**Question 3.** Evaluate the following definite integrals:

(a) *(5 marks)* $\int_0^3 x\sqrt{x+1}\,dx$

LET $u = x+1 \Rightarrow u-1 = x$

$du = dx$

IF $x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 4$

$$= \int_1^4 (u-1)\sqrt{u}\,du = \int_1^4 \left(u^{3/2} - u^{1/2}\right) du$$

$$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \left[\frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2}\right] - \left[\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right]$$

$$= \frac{2}{5}(32) - \frac{2}{3}(8) - \frac{2}{5} + \frac{2}{3} = \frac{62}{5} - \frac{14}{3}$$

$$= \frac{116}{15}$$

(b) *(4 marks)* $\int_0^{\pi/8} \sec^2 2x\,dx$

LET $u = 2x$

$du = 2dx \Rightarrow \frac{du}{2} = dx$

$\frac{du}{2} = dx$

IF $x = 0 \Rightarrow u = 0$

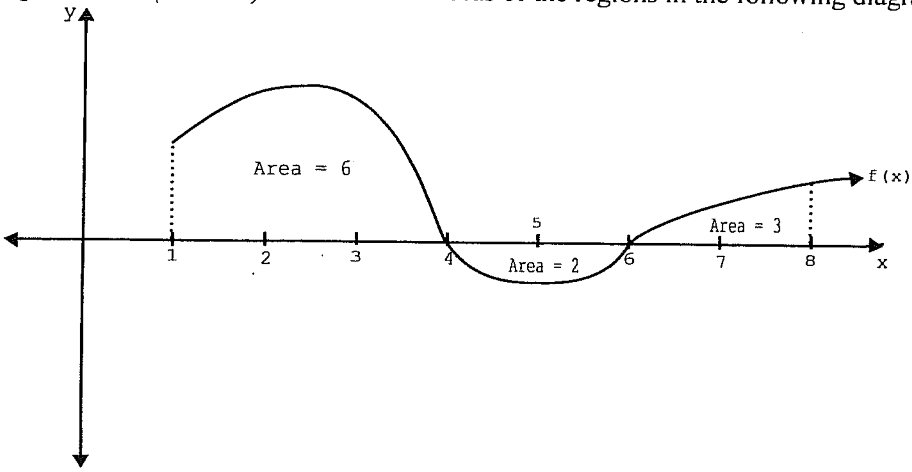
$x = \pi/8 \Rightarrow u = \pi/4$

$$= \int_0^{\pi/4} \sec^2 u \,\frac{du}{2} = \frac{1}{2}\tan u \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\tan \pi/4 - \frac{1}{2}\tan 0$$

$$= \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$$

**Question 4.** *(5 marks)* Given that the areas of the regions in the following diagram are




find the following (show your work):

(a) $\int_1^8 f(x)\,dx$ $= \int_1^4 f(x)dx + \int_4^6 f(x)dx + \int_6^8 f(x)dx$

$= 6 + (-2) + (3) = 7$

(b) $\int_1^4 f(x)\,dx - \int_4^6 f(x)\,dx + \int_6^8 f(x)\,dx$

$= 6 - (-2) + 3 = 11$

(c) $\int_8^6 f(x)\,dx$ $= -\int_6^8 f(x)dx = -(3) = -3$

(d) $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx$ $= \int_1^6 f(x)dx$

$= \int_1^4 f(x)dx + \int_4^6 f(x)dx$

$= 6 + (-2)$

$= 4$

**Question 5.** *(5 marks)* Find the average value of

$$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$$

over the interval $[1,9]$ Write you answer as a fraction.

AVE VALUE OF $f(x)$ ON $[1,9]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$= \frac{1}{9-1}\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

$$= \frac{1}{8}\int_1^9 \left(x^{1/2} + x^{-1/2}\right) dx$$

$$= \frac{1}{8}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_1^9$$

$$= \frac{1}{8}\left[\left(\frac{2}{3}(9)^{3/2} + 2(9)^{1/2}\right) - \left(\frac{2}{3}(1)^{3/2} + 2(1)^{1/2}\right)\right]$$

$$= \frac{1}{8}\left[\frac{2}{3}(27) + 2(3) - \frac{2}{3} - 2\right]$$

$$= \frac{1}{8}\left[\frac{52}{3} + 4\right]$$

$$= \frac{1}{8}\left(\frac{64}{3}\right)$$

$$= \frac{8}{3}$$

**Question 6.** *(6 marks)* Find the area of the region bounded by $f(x) = x^3 + x^2 - 3x + 1$ and $g(x) = 1 - x$. Sketch a graph to help solve the problem.

<u>INTERSECTION</u>

$f(x) = g(x)$

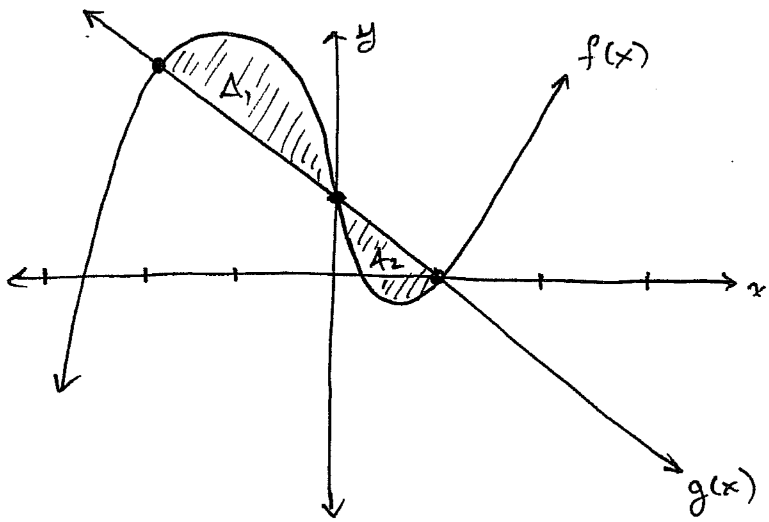
$x^3 + x^2 - 3x + 1 = 1 - x$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0$

$\therefore x = -2, 0, 1$




$\therefore A = A_1 + A_2$

$$= \int_{-2}^{0} [f(x) - g(x)]dx + \int_0^1 [g(x) - f(x)]dx$$

$$= \int_{-2}^{0} [(x^3 + x^2 - 3x + 1) - (1 - x)]dx + \int_0^1 [(1-x) - (x^3 + x^2 - 3x + 1)]dx$$

$$= \int_{-2}^{0} (x^3 + x^2 - 2x)dx + \int_0^1 (2x - x^2 - x^3)dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \left[(0 + 0 - 0) - \left(\frac{(-2)^4}{4} + \frac{(-2)^3}{3} - (-2)^2\right)\right] + \left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - (0 - 0 - 0)\right]$$

$$= -4 + \frac{8}{3} + 4 + 1 - \frac{1}{3} - \frac{1}{4}$$

$$= 1 + \frac{7}{3} - \frac{1}{4} = \frac{37}{12}$$