Dawson College: **Calculus II - Commerce**: 201-NYB-05 S13 May 3, 2012

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 3

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

Question 1. *(5 marks)* The supply function for the a product is given by

$$p = \sqrt{0.01x^2 + 48}$$

where p is the unit wholesale price in dollars and x stands for the quantity that will be made available in the market by the supplier, measured in units of a thousand. Use Simpson's rule with $n = 4$ to estimate the producer's surplus if the market equilibrium quantity suplied is 10 000 ($\bar{x} = 10$). (You may use decimals).

$$\left(\text{PS} = \bar{p}\bar{x} - \int_0^{\bar{x}} S(x)\,dx\right)$$

$\Delta x = \frac{10-0}{4} = 2.5$

$x_0 = 0, \quad x_1 = 2.5, \quad x_2 = 5$

$x_3 = 7.5 \quad x_4 = 10$

$\bar{p} = \sqrt{0.01(10)^2 + 48} = 7$

$\therefore \text{PS} = 7(10) - \int_0^{10} \sqrt{0.01x^2+48}\,dx$

$\approx 70 - \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$= 70 - \frac{2.5}{3}\left[f(0) + 4f(2.5) + 2f(5) + 4f(7.5) + f(10)\right]$

$= 70 - \frac{2.5}{3}\left[\sqrt{48} + 4\sqrt{0.01(2.5)^2+48} + 2\sqrt{0.01(5)^2+48} + 4\sqrt{0.01(7.5)^2+48} + \sqrt{0.01(10)^2+48}\right]$

$= 70 - 0.8\bar{3}\left[83.4262162\right]$

$= 70 - 69.5218$

$\doteq 0.47815$

$\therefore$ THE PRODUCER'S SURPLUS IS \$478.15

Question 2. *(5 marks)* Evaluate the following improper integral. Remember to use **correct notation**.

$$\int_0^{\infty} xe^{-x+1}dx = \lim_{b\to\infty} \int_0^b x e^{-4x}\, dx$$

$$\int x e^{-4x}\, dx \qquad \text{LET} \quad u = x \quad du = dx \qquad dv = e^{-4x}\, dx \quad v = -\frac{e^{-4x}}{4}$$

$$= -\frac{xe^{-4x}}{4} + \frac{1}{4}\int e^{-x+1}\, dx$$

$$= -\frac{xe^{-4x}}{4} - \frac{1}{16}e^{-x+1} + C$$

$$\therefore \quad I = \lim_{b\to\infty} \left[-\frac{xe^{-4x}}{4} - \frac{e^{-4x}}{16} \right]_0^b$$

$$= \lim_{b\to\infty} \left[-\frac{be^{-4b}}{4} - \frac{e^{-4b}}{16} + 0 + \frac{e^0}{16} \right]$$

$$= \lim_{b\to\infty} \left[-\frac{b}{4} e^{-4b} - \frac{e^{-4b}}{16} + \frac{1}{16} \right]$$

NOW, $\lim_{b\to\infty} -be^{-4b} = \lim_{b\to\infty} \frac{-b}{e^{4b}} = \text{I.F. } \frac{\infty}{\infty}$

$$\stackrel{H}{=} \lim_{b\to\infty} \frac{-1}{4e^{4b}} = 0$$

$$\therefore \quad I = 0 - 0 + \frac{1}{16}$$

$$= 16$$

Question 3. *(5+2 marks)*
(a) Solve the following differential equation. (Solve for y in your final answer).

$$y' = \frac{xy}{\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{xy}{\sqrt{x^2+1}} \Rightarrow \frac{1}{y}dy = \frac{x}{\sqrt{x^2+1}} \Rightarrow \int \frac{1}{y}dy = \underbrace{\int \frac{x}{\sqrt{x^2+1}}dx}_{I}$$

LET $u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$\Rightarrow I = \int \frac{x}{\sqrt{u}} \frac{du}{2x} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2} \cdot 2u^{1/2} + c$$

$$= \sqrt{x^2+1} + c$$

$$\therefore \ln|y| = \sqrt{x^2+1} + c$$

$$e^{\ln|y|} = e^{\sqrt{x^2+1} + c}$$

$$|y| = e^{\sqrt{x^2+1}} e^{c}$$

$$\therefore |y| = |C_1| e^{\sqrt{x^2+1}}$$

$$y = \pm |C_1| e^{\sqrt{x^2+1}} \Rightarrow y = C_1 e^{\sqrt{x^2+1}}$$

(b) Find the particular solution of the above differential equation that satisfies the initial condition $y(0) = 1$. (do not use decimals)

$$1 = C_1 e^{\sqrt{0^2+1}}$$

$$\frac{1}{e} = C_1$$

$$\therefore y = \frac{1}{e} e^{\sqrt{x^2+1}}$$

Question 4. *(4+3 marks)*
(a) Find the third Taylor polynomial of $f(x)=\sqrt{\ -x}$ at $x=0$ (do not use decimals)

$$f(x)=\sqrt{1-x} \qquad \Rightarrow \qquad f(0)=1$$

$$f'(x)=-\frac{1}{2}(1-x)^{-1/2} \qquad f'(0)=-1/2$$

$$f''(x)=-\frac{1}{4}(1-x)^{-3/2} \qquad f''(0)=-1/4$$

$$f'''(x)=-\frac{3}{8}(1-x)^{-5/2} \qquad f''(0)=-3/8$$

$$\therefore\ T_3(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3$$

$$=1-\frac{1}{2}x-\frac{1}{4}\cdot\frac{1}{2}x^2-\frac{3}{8}\cdot\frac{1}{6}x^3$$

$$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$$

(b) Use the third ($n=3$) Taylor polynomial to approximate

$$\int_0^{0.1} x^2\sqrt{4-x}\,dx \qquad \approx \int_0^{0.1} x^2\, T_3(x)\,dx$$

(you may use decimals).

$$=\int_0^{0.1} x^2\left[1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\right]dx$$

$$=\int_0^{0.1} x^2-\frac{1}{2}x^3-\frac{1}{8}x^4-\frac{1}{16}x^5\,dx$$

$$=\left[\frac{x^3}{3}-\frac{1}{8}x^4-\frac{1}{40}x^5-\frac{1}{96}x^6\right]_0^{0.1}$$

$$=\frac{0.1^3}{3}-\frac{1}{8}(0.1)^4-\frac{1}{40}(0.1)^5-\frac{1}{96}(0.1)^6-0$$

Question 5. *(2 marks)* Find the nth term if the following sequence

$$\frac{1}{1}, \ -\frac{1}{2}, \ \frac{1}{6}, \ -\frac{1}{24}, \ \frac{1}{120}, \ -\frac{1}{720}, \ \ldots$$

$$a_n = \frac{(-1)^{n+1}}{n!}$$

Question 6. *(2+3+3 marks)* Determine if the following sequences converge or diverge. If a sequence converges find the limit of the sequence.

a) $a_n = \dfrac{n!}{5n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)(\cancel{n})}{5\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)}{5}$$

$$= \infty \quad \therefore \text{ THE LIMIT DIVERGES}$$

b) $a_n = \dfrac{\ln(n)}{3n^2+2n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n)}{3n^2+2n} = \text{I.F } \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{\frac{1}{n}}{6n+2}$$

$$= \lim_{n\to\infty} \frac{1}{6n^2+2n} = 0 \qquad \text{(CONVERGES)}$$

c) $a_n = \dfrac{4^{n+1}}{3+4^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^{n+1}}{3+4^n} = \lim_{n\to\infty} \frac{\frac{4^{n+1}}{4^n}}{\frac{3}{4^n}+\frac{4^n}{4^n}}$$

$$= \lim_{n\to\infty} \frac{4}{3/4^n+1} = \frac{4}{0+1} = 4$$

Question 7. *(5+5 marks)* Determine if the following series converge or diverge. If a series converges find the sum.

a) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+2}\right) \;=\; \sum_{n=1}^{\infty} \left[\ln(n) - \ln(n+2)\right]$

$S_n = (\ln 1 - \cancel{\ln 3}) + (\ln 2 - \cancel{\ln 4}) + (\cancel{\ln 3} - \cancel{\ln 5})$

$+ (\cancel{\ln 4} - \cancel{\ln 6}) + \dots + (\ln(\cancel{n-3}) - \ln(\cancel{n-1}))$

$+ (\ln(\cancel{n-2}) - \ln(\cancel{n})) + (\ln(\cancel{n-1}) - \ln(n+1))$

$+ (\ln(\cancel{n}) - \ln(n+2))$

$= \ln 1 + \ln 2 - \ln(n+1) - \ln(n+2)$

$= \ln 2 - \ln(n+1) - \ln(n+2)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\ln 2 - \ln(n+1) - \ln(n+2)\right]$

$= \quad -\infty$

$\therefore$ THE SERIES DIVERGES

b) $\displaystyle\sum_{n=2}^{\infty}\frac{2+3^n}{4^{n-2}} = \sum_{n=2}^{\infty}\left(\frac{2}{4^{n-2}}+\frac{3^n}{4^{n-2}}\right) = \delta$

$$\sum_{n=2}^{\infty}\frac{2}{4^{n-2}} = \sum_{n=2}^{\infty}\frac{2}{4^{n-1}} = \sum_{n=2}^{\infty}2\left(\frac{1}{4}\right)^{n-1} = \frac{2}{1-\frac{1}{4}} = \frac{2}{3/4} = \frac{8}{3}$$

GEOMETRIC WITH $a=2$, $r=\frac{1}{4}$

SO $|r|=\frac{1}{4}<1 \Rightarrow$ CONVERGES

$$\sum_{n=2}^{\infty}\frac{3^n}{4^{n-2}} = \sum_{n=1}^{\infty}\frac{3^{n+1}}{4^{n-1}} = \sum_{n=1}^{\infty}3^2\left(\frac{3^{n-1}}{4^{n-1}}\right) = \sum_{n=1}^{\infty}9\left(\frac{3}{4}\right)^{n-1}$$

GEOMETRIC WITH $a=9$, $r=\frac{3}{4}$

$\therefore$ $|r|=\frac{3}{4}<1 \Rightarrow$ CONVERGES

$$\therefore \quad \sum_{n=1}^{\infty}9\left(\frac{3}{4}\right)^{n-1} = \frac{9}{1-\frac{3}{4}} = 36$$

$$\text{SO} \quad \delta = \sum_{n=2}^{\infty}\frac{2}{4^{n-2}} + \sum_{n=2}^{\infty}\frac{3^n}{4^{n-2}}$$

$$= \frac{8}{3}+36$$

$$= \frac{116}{3}$$