**Dawson College: Calculus II - Commerce:** 201-NYB-05 S14

May 7, 2012

**Last Name:** SOLUTIONS

**First Name:** ____________

**Student ID:** ____________

# Test 3

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

**Question 1.** *(4 marks)* Use the trapezoidal rule with $n = 4$ to approximate the following integral

$$\int_0^3 \frac{\ln(x+1)}{x+1}\,dx$$

(you may use decimals).

$$\Delta x = \frac{3-0}{4} = \frac{3}{4} = 0.75$$

$x_0 = 0$, $x_1 = 0.75$, $x_2 = 1.5$, $x_3 = 2.25$, $x_4 = 3$

$$\therefore \int_0^3 \frac{\ln(x+1)}{x+1}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{0.75}{2}\left[\frac{\ln(1)}{1} + 2\cdot\frac{\ln(1.75)}{1.75} + 2\cdot\frac{\ln(2.5)}{2.5} + 2\cdot\frac{\ln(3.25)}{3.25} + \frac{\ln(4)}{4}\right]$$

$$= 0.375\left[0 + 0.6395609005 + 0.7330325855 + 0.7253261516 + .3465735903\right]$$

$$= 0.375(2.44449322 8)$$

$$= 0.916684961$$

**Question 2.** *(5 marks)* Find the area under the function $f(x) = xe^{-5x}$ and above the $x$-axis on the interval $[0, \infty)$. Remember to use **correct notation.** (Do not use decimals).

$$\text{AREA} = \int_0^\infty xe^{-5x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-5x}\,dx$$

$$\int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} + \frac{1}{5}\int e^{-5x}\,dx$$

$$\boxed{\begin{array}{ll} \text{LET } u = x & dv = e^{-5x}dx \\ du = dx & v = \frac{e^{-5x}}{-5} \end{array}}$$

$$= -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

$$\therefore \text{AREA} = \lim_{b\to\infty}\left[-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x}\right]_0^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{5}be^{-5b} - \frac{1}{25}e^{-5b} + 0 + \frac{1}{25}e^0\right]$$

$$\lim_{b\to\infty} be^{-5b} = \lim_{b\to\infty}\frac{b}{e^{5b}} = \text{I.F. } \frac{\infty}{\infty} = \lim_{b\to\infty}\frac{1}{5e^{5b}} = 0$$

$$\therefore \text{AREA} = \left[0 - 0 + 0 + \frac{1}{25}\right]$$

$$= \frac{1}{25}$$

**Question 3.** *(5+2 marks)*
**(a)** Solve the following differential equation. (Solve for $y$ in your final answer).

$$y' = \frac{x}{x^2y+y} \Rightarrow \frac{dy}{dx} = \frac{x}{y(x^2+1)}$$

$$\Rightarrow y\,dy = \frac{x}{x^2+1}\,dx \Rightarrow \int y\,dy = \int \frac{x}{x^2+1}\,dx$$

$$\frac{y^2}{2} = \frac{1}{2}\ln(x^2+1) + c$$

$$y^2 = \ln(x^2+1) + C_1$$

$$y = \pm\sqrt{\ln(x^2+1)+C_1}$$

SET $u = x^2+1$

$du = 2x\,dx$

$\frac{du}{dx} = dx$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\frac{du}{2x}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$

$$= \frac{1}{2}\ln|x^2+1| + c$$

**(b)** Find the particular solution of the above differential equation that satisfies the initial condition $y(0) = -2$. (do not use decimals)

$$-2 = -\sqrt{\ln(0^2+1)+C_1}$$

$$2 = \sqrt{C_1}$$

$$4 = C_1$$

$$\therefore y = -\sqrt{\ln(x^2+1)+4}$$

**Question 4.** *(4+2 marks)*
**(a)** Find the third Taylor polynomial of $f(x) = \sqrt{2x+1}$ at $x = 0$ (do not use decimals)

$f(x) = \sqrt{2x+1}$ $\qquad\qquad f(0) = 1$

$f'(x) = 2 \cdot \frac{1}{2}(2x+1)^{-1/2} = (2x+1)^{-1/2}$ $\qquad\qquad f'(0) = 1$

$f''(x) = -\frac{1}{2} \cdot 2(2x+1)^{-3/2} = -(2x+1)^{-3/2}$ $\qquad \Rightarrow \quad f''(0) = -1$

$f'''(x) = \frac{3}{2}(2)(2x+1)^{-5/2} = 3(2x+1)^{-5/2}$ $\qquad\qquad f'''(0) = 3$

$\therefore \; T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$

**(b)** Use this Taylor polynomial to approximate $\sqrt{1.5}$ (you may use decimals).

$f(x) = \sqrt{2x+1} = \sqrt{1.5}$

$\Rightarrow 2x+1 = 1.5$

$\Rightarrow 2x = 0.5$

$\Rightarrow x = 0.25$

$= f(0.25) \approx T_3(0.25) = 1 + 0.25 - \frac{1}{2}(0.25)^2 + \frac{1}{2}(0.25)^3$

$= 1.22656$

b) $\sum_{n=2}^{\infty} \frac{2^{n-2}+3}{5^n} = \sum_{n=2}^{\infty}\left(\frac{2^{n-2}}{5^n} + \frac{3}{5^n}\right)$

$$\sum_{n=2}^{\infty} \frac{2^{n-2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{5^2 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{25}\left(\frac{2}{5}\right)^{n-1}$$

GEOMETRIC, $a = \frac{1}{25}$, $r = \frac{2}{5}$, $|r| = \frac{2}{5} < 1$ (CONVERGES)

$$\therefore \sum_{n=1}^{\infty} \frac{1}{25}\left(\frac{2}{5}\right)^{n-1} = \frac{1/25}{1 - 2/5} = \frac{1/25}{3/5} = \frac{1}{25} \cdot \frac{5}{3} = $$

$$= \frac{1}{15}$$

$$\sum_{n=2}^{\infty} \frac{3}{5^n} = \sum_{n=1}^{\infty} \frac{3}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3}{25}\left(\frac{1}{5}\right)^{n-1} = \frac{3/25}{1 - 1/5} = \frac{3/25}{4/5} = \frac{3}{20}$$

GEOMETRIC, $a = 3$, $r = \frac{1}{5}$

$|r| = \frac{1}{5} < 1$ (CONVERGES)

$$\therefore \sum_{n=2}^{\infty} \frac{2^{n-2}+3}{5^n} = \sum_{n=2}^{\infty} \frac{2^{n-1}}{5^n} + \sum_{n=2}^{\infty} \frac{3}{5^n}$$

$$= \frac{1}{15} + \frac{3}{20}$$

$$= \frac{13}{60}$$

**Question 5.** *(2 marks)* Find the $n$th tern if the following sequence

$$3, \quad -1, \quad \frac{1}{3}, \quad -\frac{1}{9}, \quad \frac{1}{27}, \quad -\frac{1}{81}, \quad \ldots$$

$$= 3^{1}, \ -3^{0}, \ 3^{-1}, \ -3^{-2}, \ 3^{-3}, \ -3^{-4}, \ldots$$

$$\therefore a_n = (3)^{-n+2}(-1)^{n+1} \text{ OR } \frac{(-1)^{n+1}3^2}{3^n} \text{ OR } \frac{(-1)^{n+1}}{3^{n-2}}$$

**Question 6.** *(2+4+3 marks)* Determine if the following sequences converge or diverge. If a sequence converges find the limit of the sequence. (Show all of your work).

a) $a_n = \dfrac{7n}{n!}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7n}{n!} = \lim_{n\to\infty} \frac{7n}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n} = \lim_{n\to\infty} \frac{7}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)}$$

$$= \lim_{n\to\infty} \frac{7}{(n-1)!} = 0 \quad \text{(CONVERGES)}$$

**b)** $a_n = \dfrac{e^n + n}{3n^2 + 1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n + n}{3n^2 + 1} = \text{I.F. } \frac{\infty}{\infty} \overset{H}{=} \lim_{n\to\infty} \frac{e^n + 1}{6n}$$

$$= \text{I.F. } \frac{\infty}{\infty} \overset{H}{=} \lim_{n\to\infty} \frac{e^n}{6} = \infty \qquad \text{(DIVERGES)}$$

**c)** $a_n = \dfrac{5^{n+1}}{2 + 5^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^{n+1}}{2+5^n} = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{5^n}}{\frac{2}{5^n} + \frac{5^n}{5^n}} = \lim_{n\to\infty} \frac{5}{\frac{2}{5^n} + 1}$$

$$= \frac{5}{0+1} = 5 \qquad \text{(CONVERGES)}$$

**Question 7.** *(6+5 marks)* Determine if the following series converge or diverge. If a series converges find the sum.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{(n+2)n}$

$\left(\text{Hint: use partial fractions to write } \quad \dfrac{-2}{(n+2)n} = \dfrac{A}{n+2} + \dfrac{B}{n} \quad \text{ and solve for } A \text{ and } B.\right)$

$$\frac{-2}{(n+2)n} = \frac{A}{n+2} + \frac{B}{n}$$

$$-2 = An + B(n+2)$$

F $n=0$ | IF $n=-2$
---|---
$-2 = 2B$ | $-2 = -2A$
$-1 = B$ | $1 = A$

$$\therefore \sum_{n=1}^{\infty} \frac{-2}{(n+2)n} = \sum_{n=1}^{\infty}\left[\frac{1}{n+2} - \frac{1}{n}\right]$$

$$S_n = \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{1}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{2}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{6}} - \cancel{\tfrac{1}{4}}\right)$$
$$+ \dots + \left(\cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n-3}}\right) + \left(\cancel{\tfrac{1}{n}} - \cancel{\tfrac{1}{n-2}}\right) + \left(\tfrac{1}{n+1} - \cancel{\tfrac{1}{n-1}}\right)$$
$$+ \left(\tfrac{1}{n+2} - \cancel{\tfrac{1}{n}}\right)$$

$$= -1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$= -1 - \frac{1}{2} + 0 + 0$$

$$= -3/2$$

$$\therefore \sum_{n=1}^{\infty} \frac{-2}{(n+2)n} = -\frac{3}{2}$$ (THE SERIES CONVERGES)