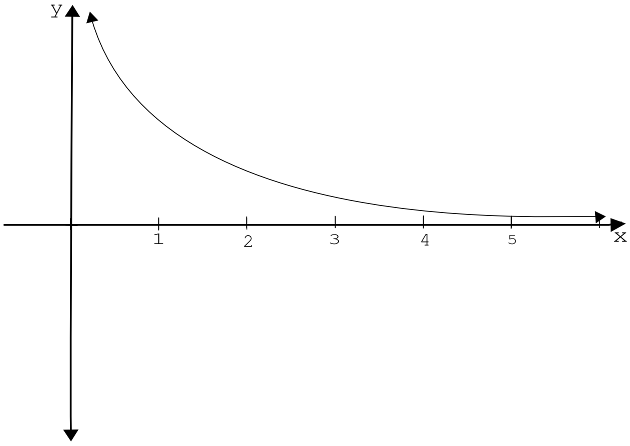
Improper Integrals

Lets look at the graph of $f(x) = \frac{1}{x^2}$ on $(0, \infty)$



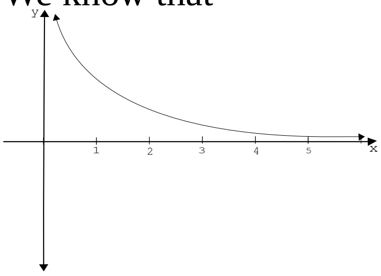
Even though the function continues on indefinitely, the area R might be finite.

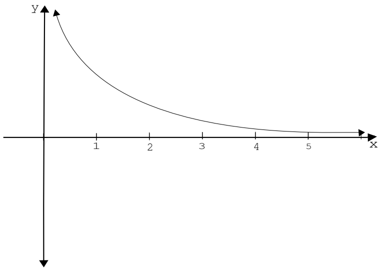
Using our notation we would like to say

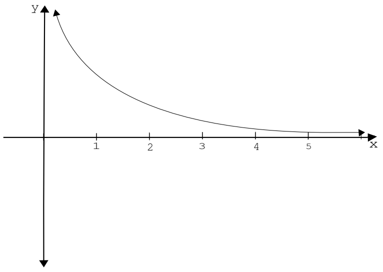
$$R = \int_1^\infty \frac{1}{x^2}\,dx$$

but what does this mean?

We know that







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This leads us to the following definition:

Definition Let f be a continuous function on the unbounded interval $[a, \infty)$. Then the improper integral of f over $[a, \infty)$ is defined by

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

if this limit exists.

Examples: Evaluate the following.

a) $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

b) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$

c) $\int_0^{\infty} xe^{-x/2}\,dx$

Similarly, we can define.

Definition Let f be a continuous function on the unbounded interval $(-\infty, b]$. Then the improper integral of f over $(-\infty, b]$ is defined by

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$$

if this limit exists.

Example: Find the area bounded above by the x-axis, below by the curve $y = -e^{x/3}$ and on the right by $y = 1$.

Definition Let f be a continuous function over the unbounded interval $(-\infty, \infty)$. Then the improper integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

and is convergent if both integrals on the right are convergent. If **one** of the integrals on the right is divergent then we say that the integral is divergent.

Example: Evaluate.

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

Example: Evaluate.

$$\int_{-\infty}^{\infty} x^3\, dx$$

We recently saw that the present value of an annuity is given by

$$PV = mP\int_0^T d^{-rt}\, dt = mP(1 - e^{-rT})$$

If the payents of an annuity are allowed to coninue indefinitey, we have what is called a **perpetuity**. We can find the present value of a perpetuity by taking

$$PV = mP\int_0^\infty e^{-rt}\, dt$$

The Present Value of a Perpetuity

The present value of a perpetuity is given by

$$PV = \frac{mP}{r}$$

where m is the number of payments per year, P is the size of each payment, and r is the interest rate (compounded continuously).

Example: The Robinson family wishes to create a scholarship fund at a college. If a scholarship in the amount of $5000 is awarded annually beginning 1 year from now, find the amount of the endowment they are required to make now. assume that this fund will earn interest at a rate of 8% per year compounded continuously.