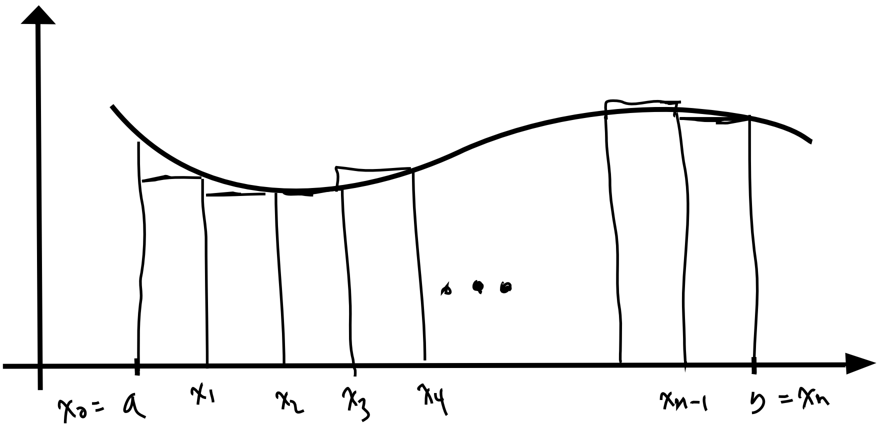
# The Riemann Sum and the Definite Integral

Let's apply this idea to use approximating rectangles fing the area of the region $S$ under a positive function $f(x)$ on the interval $[a,b]$.




The width of the interval $[a,b]$ is $b-a$ so the width of each strip is

$$\Delta x = \frac{b-a}{n}$$

The strips divide the interval into $n$ subintervals $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, $\ldots$ $[x_{n-1},x_n]$. The endpoints are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$$

$$x_i = a + i\Delta x, \ldots, \quad x_n = a + n\Delta x$$

Using the right endpoints we approximate the area with

$$R_n = \Delta x f(x_1) + \Delta x f(x_2) + \ldots + \Delta x f(x_n)$$

This sum of the areas of approximating rectangles is called a **Riemann Sum**.
As we saw last class, the approximation gets better as $n$ increases. Because this is the case we define the following:

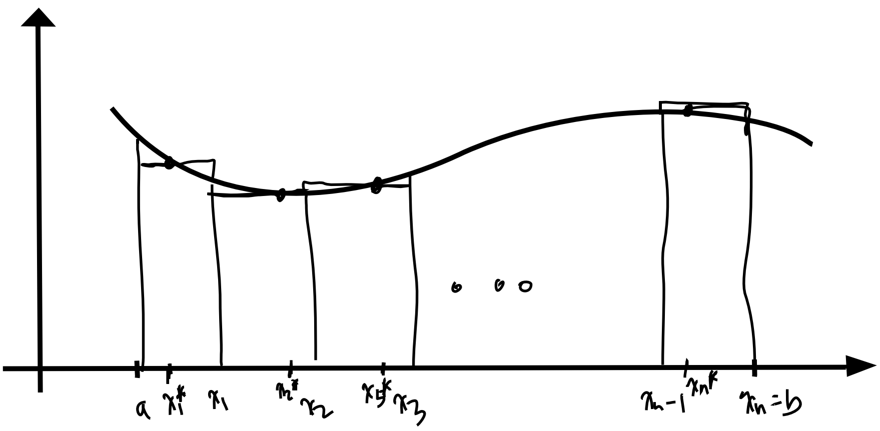
**Definition:** The **Area** of the region $S$ that lies under the graph of a positive continuous function $f$ is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x]$$

It can be shownthat this limit always exists (for a continuous function) and we get the same value if we use

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

In fact, instead of using the left or the right endpoints for the heights of the rectangles we can pick any point we want $x_i*$ from each of the subintervals $[x_{i-1}, x_i]$.




Still

$$A = \lim_{n\to\infty} \left[\Delta x f(x_1^*) + \Delta x f(x_2^*) + \ldots + \Delta x f(x_n^*)\right]$$

Notice that we can write this definition using sigma notation:

$$\begin{aligned} A = \lim_{n\to\infty} R_n &= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

This limit is called the **definite integral**.

**Definition:** If $f(x)$ is a continuous function defined on $[a,b]$, and if $[a,b]$ is divided into $n$ equal subintervals of width $\Delta x = \frac{b-a}{n}$, and if $x_i = a + i\Delta x$ is the right endpoint of the subinterval $[x_{i-1}, x_i]$ then the **definite integral** of $f$ from $a$ to $b$ is the number

$$\int_{\substack{a \\ \uparrow \\ \text{START}}}^{b \,\leftarrow\, \text{END}} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example: Use the definition of the definite integral to evaluate the following

**1)** $\int_0^3 (x-2x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ ← b, ← a

- $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$
- $x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$
- $f(x_i) = x_i - 2(x_i)^2 = \frac{3i}{n} - 2\left(\frac{3i}{n}\right)^2 = \frac{3i}{n} - 2\left(\frac{9i^2}{n^2}\right) = \frac{3i}{n} - \frac{18i^2}{n^2}$
- $f(x_i)\Delta x = \underbrace{\left[\frac{3i}{n} - \frac{18i^2}{n^2}\right]}_{f(x_i)}\underbrace{\left(\frac{3}{n}\right)}_{\Delta x} = \frac{9i}{n^2} - \frac{54i^2}{n^3}$ ← WARNING! MAKE SURE YOU USE $\Delta x$ NOT $x_i$!
- $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{9i}{n^2} - \frac{54i^2}{n^3}\right] = \sum_{i=1}^{n}\frac{9i}{n^2} - \sum_{i=1}^{n}\frac{54i^2}{n^3}$

$$= 9\sum_{i=1}^{n}\frac{i}{n^2} - 54\sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{9}{n^2}\sum_{i=1}^{n} i - \frac{54}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

- $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{9}{2}\cdot\frac{n+1}{n} - \lim_{n\to\infty} 9\cdot\frac{(n+1)(2n+1)}{n^2}$$

$$= \frac{9}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1} - 9\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}$$

$$= \frac{9}{2}\lim_{n\to\infty}\frac{1+\overset{\to 0}{1/n}}{1} - 9\lim_{n\to\infty}\frac{2+\overset{\to 0}{3/n}+\overset{\to 0}{1/n^2}}{1} = \frac{9}{2}\cdot\frac{1+0}{1} - 9\cdot\frac{2+0+0}{1}$$

$$= \frac{9}{2} - 18 = \frac{-27}{2}$$

$$\int_0^3 (x-2x^2)\,dx = -27/2$$

**2)** $\int_2^5 (8x - x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

- $\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$ • $x_i = a + i\Delta x = 2 + i\left(\frac{3}{n}\right) = 2 + \frac{3i}{n}$

- $f(x_i) = 8x_i - (x_i)^2 = 8\left(2 + \frac{3i}{n}\right) - \left(2 + \frac{3i}{n}\right)^2$

$= 16 + \frac{24i}{n} - \left(4 + \frac{12i}{n} + \frac{9i^2}{n^2}\right) = 12 + \frac{12i}{n} - \frac{9i^2}{n^2}$

- $f(x_i)\Delta x = \left[12 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]\left(\frac{3}{n}\right) = \frac{36}{n} + \frac{36i}{n^2} - \frac{27i^2}{n^3}$

- $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{36}{n} + \frac{36i}{n^2} - \frac{27i^2}{n^3}\right]$

$= \sum_{i=1}^{n} \frac{36}{n} + \sum_{i=1}^{n} \frac{36i}{n^2} - \sum_{i=1}^{n} \frac{27i^2}{n^3}$

$= \frac{36}{n}\sum_{i=1}^{n} 1 + \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{36}{n}\cdot n + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

- $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[36 + 18\cdot\frac{n+1}{n} - \frac{9}{2}\cdot\frac{2n^2+3n+1}{n^2}\right]$

$= \lim_{n\to\infty} 36 + 18\lim_{n\to\infty}\frac{1+1/n}{1} - \frac{9}{2}\lim_{n\to\infty}\frac{2+3/n+1/n^2}{1}$

$= 36 + 18\cdot\frac{1+0}{1} - \frac{9}{2}\cdot\frac{2+0+0}{1} = 45$

$\therefore \int_2^5 (8x - x^2)\,dx = 45$ !

**3)** $\int_0^2 (x^2 + 10)\, dx$

**4)** $\int_{1}^{5} (x - 4x^2)\, dx$

**5)** $\int_{-3}^{0} (4x^2 - 5x - 1)\, dx$

**6)** $\int_0^4 (x^3 - 1)\, dx$