The Distance Problem

Suppose the odometer on our car is broken and we want to estimate the distance travelled over a 30-second period. We take speedometer readings every five seconds and get

Time(s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

Approximate how far the car travels in

a) the first five seconds

b) over the entire 30 seconds

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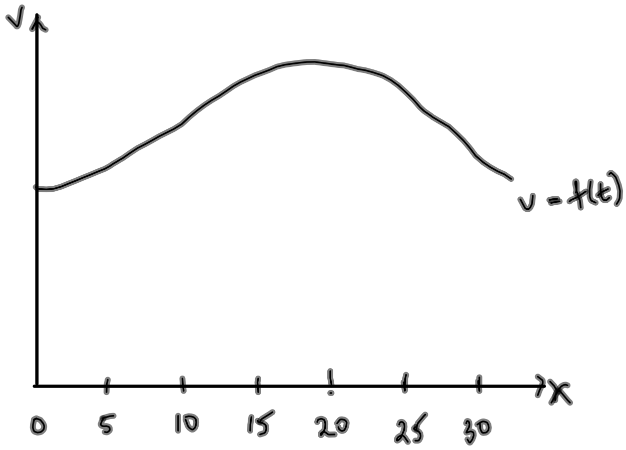
Graphically, the situation looks like



In general, suppose an object moves with velocity $v = f(t)$ where $a \le t \le b$, $f(t) \ge 0$ and we have velocity readings at times $t_0 = a,\ t_1,\ t_2,\ \ldots,\ t_n = b$ where these readings are are equally spaced. Then the time between consecutive readings is

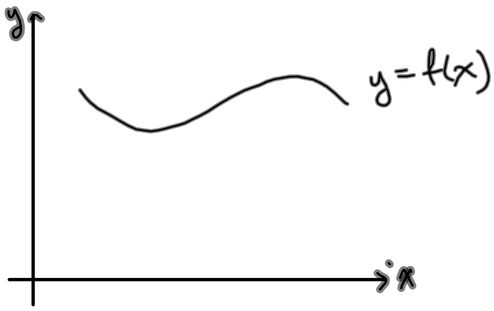
During the time interval $[t_{i-1}, t_i]$ the distance travelled is approximately . So the total distance is approximately

But the more frequent the readings, the more accurate the approximation. It seems like we could get the exact distance by taking the limit

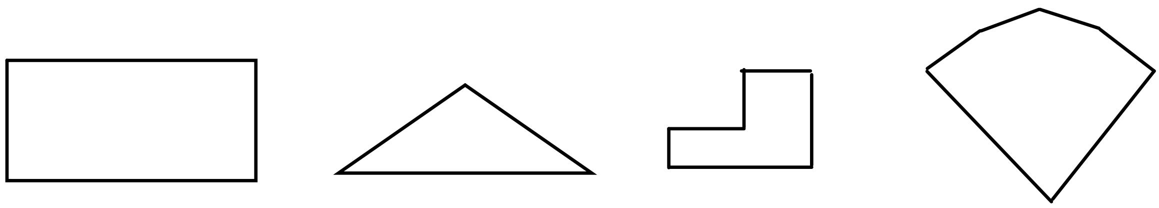
In other words the distance is given by the area under the velocity curve.

5.1 Areas and Distances

The Area Problem:

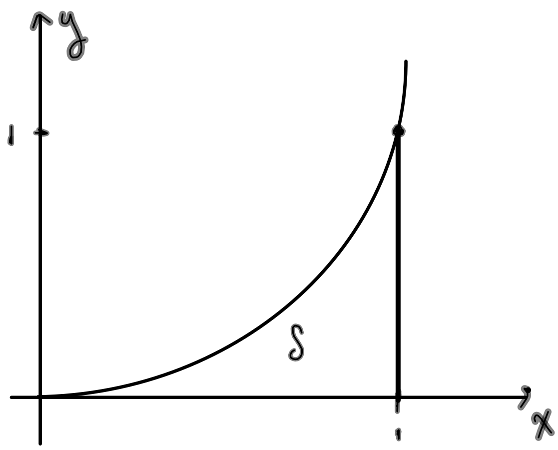


To find the Area of a Polygon we can divide the figure into smaller shapes that we can find the area of

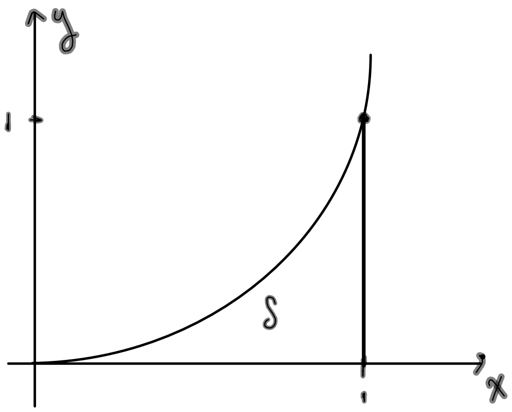


It isn't as straight forward to find the area of a region with curved sides. Part of the problem is coming up with a **precise definition of area.**

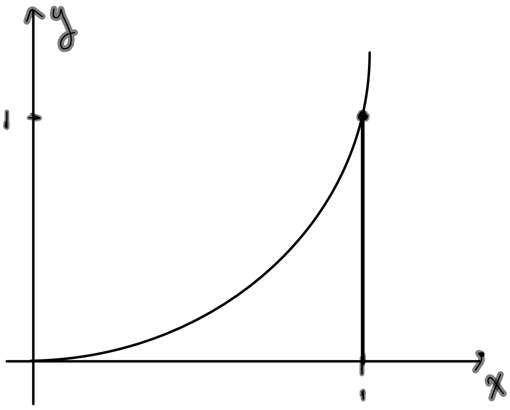
Let's start by trying to estimate the areas under $y = x^2$ from 0 to 1.



Suppose we divide S into four strips by drawing vertical lines at
$x_1 = 1/4$, $x_2 = 1/2$, $x_3 = \frac{3}{4}$ AND $x_4 = 1$



Then we can approximate the area of each strip using a rectangle



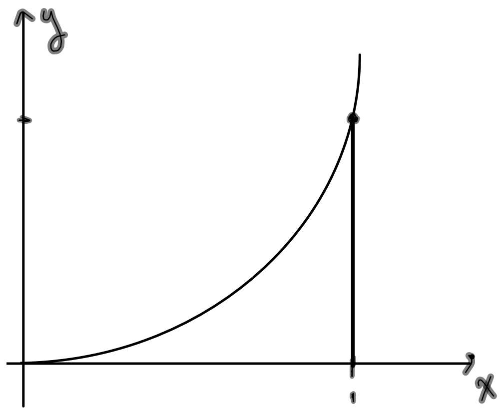
The width of each rectangle will be

and the heights will be given by the value of f at x_1 , x_2 , x_3 AND x_4

So an approximation for the area of S would be

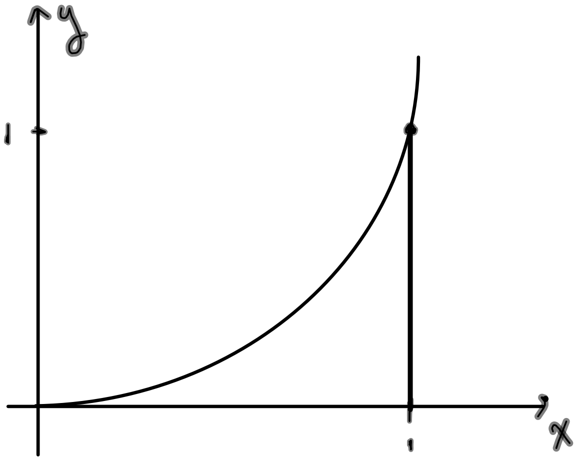
If we call A the area of S we see that

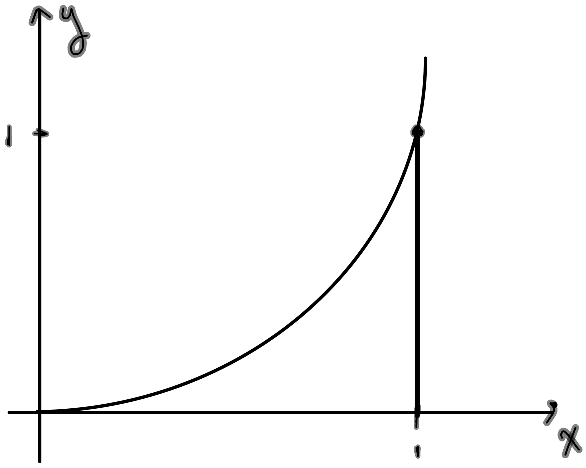
But we could have instead used the values of f at the **left endpoint** instead of the right endpoint to get the heights of the rectangles.



If we use a larger number of strips it looks like our approximation of A gets better.

Using 8 strips:





We get better estimates by increasing the number of rectangles:

http://mathworld.wolfram.com/RiemannSum.html

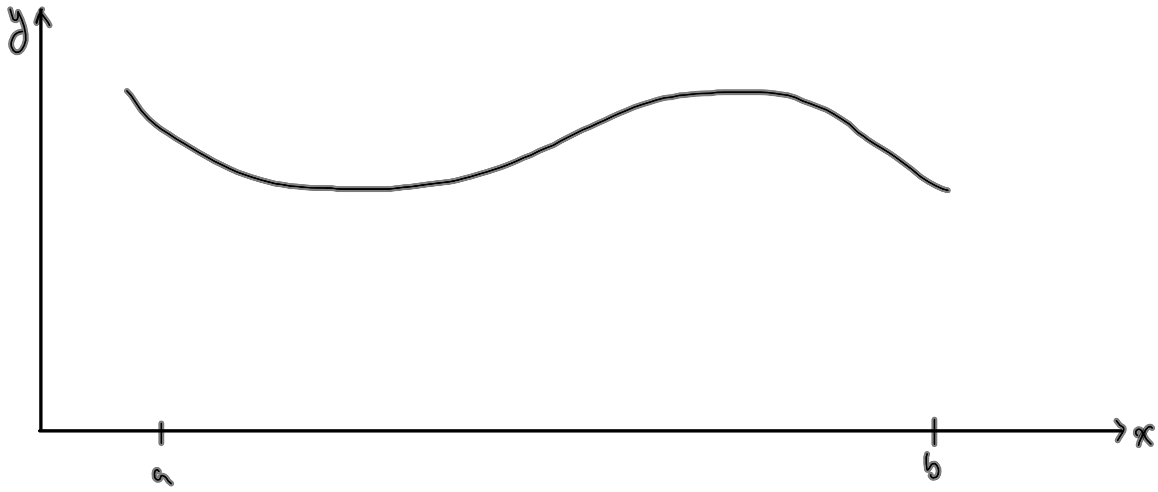
The following table shows the results of using n strips

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

It looks like L_n and R_n are approaching as n increases. It also looks like we are looking at the limit of L_n and R_n as $n \to \infty$

Lets try and write a formula for R_n using an arbitrary number of rectangles n.

Let's apply this idea to a more general region, S



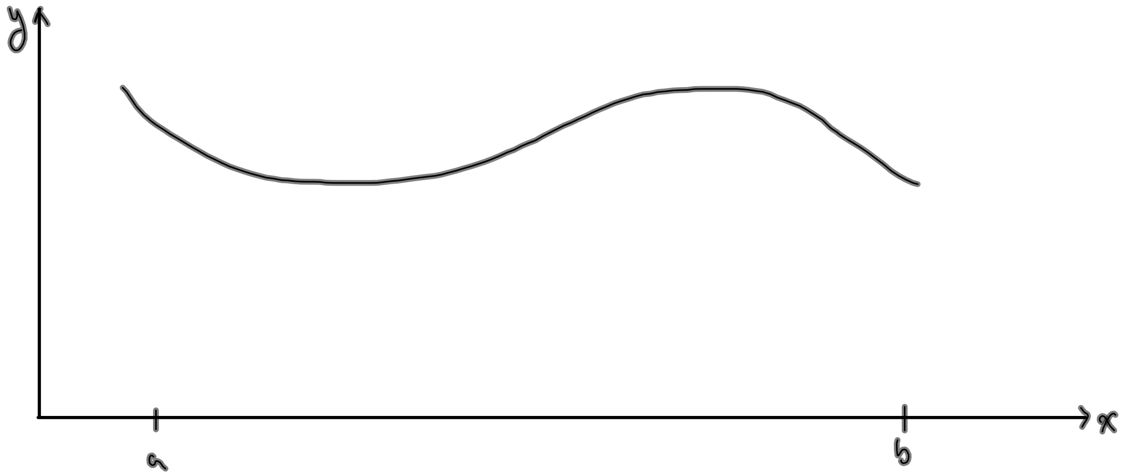
Definition: The **Area** of the region S that lies under the graph of a positive continuous function is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{x\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

It can be shown that this limit always exists for a continuous function and we get the same values if we use

$$A = \lim_{x\to\infty} L_n = \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

In fact, instead of using left or right endpoints to get the heights of the ith rectangle, we can use $f(x_i^*)$ where x_i^* is any number in the interval $[x_{i-1}, x_i]$



We call $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ the sample points. And so

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

We can write these sums using sigma notation:

$$A = \lim_{x\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

$$A = \lim_{x\to\infty} L_n = \lim_{n\to\infty}[f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

Ex: Let A be the area of the region that lies under the graph of $f(x)=e^{-x}$ between $x=0$ and $x=2$.

a)Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

b) Estimate the area by taking sample points to be the midpoints using four subintervals.

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