Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

For each fixed x value, the series becomes a series of constants that we can test for divergence or divergence. For example, if all of the constants $c_n = 1$ then this would be the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

We could as ourselves does this series converges if $x = \frac{1}{2}$. We would get

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$$

which is geometric with $|r| = 1/2 < 1$ so it converges.

We can see that

$$f(x) = 1 + x + x^2 + x^3 +^3 + \dots$$

is a function whose domain is all x values where this series converges. But since this is a geometric series it converges when $|x| < 1$ and diverges when $|x| \geq 1$. So

$f(x) = 1 + x + x^2 + x^3 +^3 + \dots$ with domain $(-1, 1)$.

In general,

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

is a function whose domain is the set of all x values for which the series converges.

The power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

is a power series centred at 0.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is called a **power series in** $(x - a)$, a **power series centred at a**.

Notice that this series should equal c_0 if $x = a$.

$$c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \dots = c_0 + 0 + 0 + 0 \dots$$

so we adopt the convention that $(x - a)^0 = 1$ even when $x = a$.

Also notice that this means that every power series converges at it's centre, $x = a$.

Examples:

1) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$

2) $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \dots$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n = (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \dots$

Ex: For what values of x are the following power series convergent?

1) $\sum_{n=0}^{\infty} n! x^n$

2) $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n}$

Example: Find the domain of the Bessel function of order 0 defined by

$$J_0 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1) The series converges only when $x = a$.

2) The series converges for all x.

3) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the **radius of convergence**. In case 1 we say that $R = 0$ and in case 2 we say that $R = \infty$.

Case 1:

Case 2:

Case 3:

The interval of all values of x where the series converges is called the **interval of convergence**. Notice that the theorem does not tell us what happens when $|x-a| = R$, in other words, at the endpoints of the interval. The series could converge at one, both or neither endpoint. We need to check them individually.

Example: Find the radius of convergence and the interval of convergence of the following series:

1) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

2) $$\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$$