Theorem: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof:

Theorem: The test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Examples:

1) $\sum_{n=1}^{\infty} \arctan(n)$

2) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

A **geometric series** has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots \qquad a \neq 0$$

where r is called the ratio. Note sometimes a geometric series is written as

$$\sum_{n=0}^{\infty} ar^{n}$$

Theorem: The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

Example: Are the following series convergent or divergent? If they are convergent, find the sum.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^{n-1}}$

2) $\displaystyle\sum_{n=2}^{\infty} 7\left(\frac{1}{5}\right)^{n-1}$

Theorem: If $\sum a_n$ and $\sum b_n$ are convergent series and c is a real number then the following series converge and

1) $\sum c\,a_n = c\sum a_n$

2) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

3) $\sum (a_n - b_n) = \sum a_n - \sum b_n$

Example: 3) $\sum_{n=2}^{\infty} \left(\frac{1}{2^n} - \frac{2^n}{5^{n-2}} \right)$

Note: If a series $\sum_{n=i}^{\infty} a_n$ converges (or diverges) so does $\sum_{n=k}^{\infty} a_n$ for any integers i and k.

Example: Express $0.\overline{81} = 0.818181\ldots$ as a ratio of integers.