5.2 The Definite Integral

If we divide and interval [a,b] into n smaller intervals by choosing partition points

The resulting collection of subintervals

is called a partition P of [a,b].

We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$. In other words $\Delta x_i = x_i - x_{i-1}$.

In each subinterval we choose sample points $x_1^*, x_2^*, ..., x_n^*$ with x_i^* in $[x_{i-1}, x_i]$
These sample points could be left endpoints or right endpoints or any point inside the interval.

A Riemann sum associated with a partition P and a function f(x) is:

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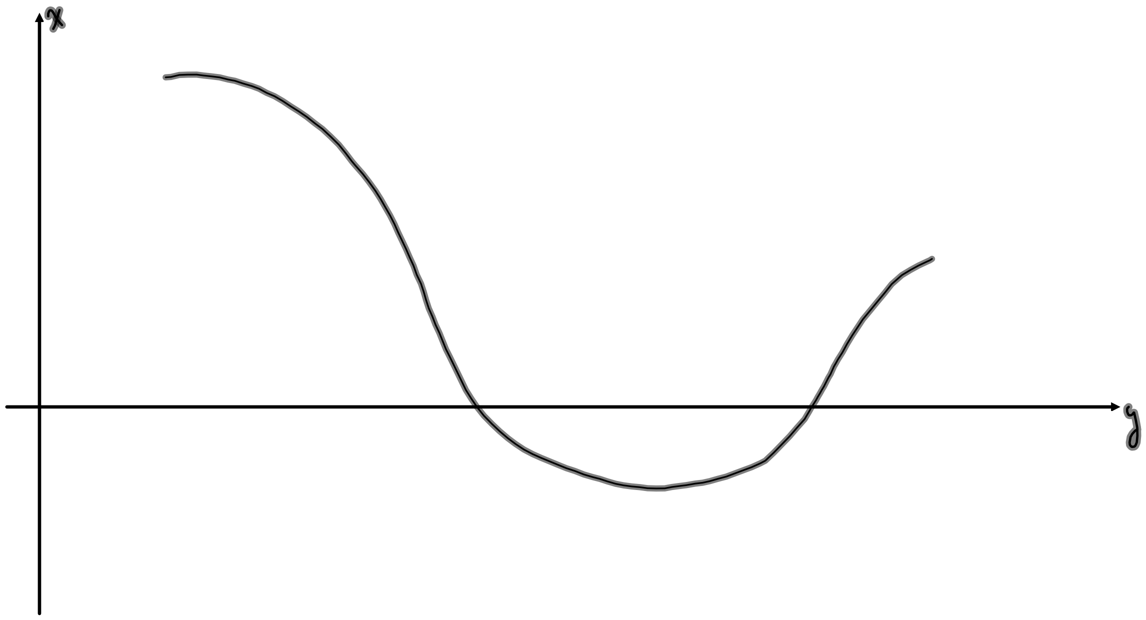
This looks like:



Notice that when $f(x_i^*)$ is negative $f(x_i^*)\Delta x$ is negative so we subtract the area of the corresponding rectangle.

(If we think of all possible partitions of [a,b] and all possible choices of sample points) we can think of taking the limit of all possible Riemann sums as n becomes large.

But since we are allowing subintervals of different lengths we need to make sure that all Δx_i approach 0. We do this by saying that the largest of these lengths, denoted $\max \Delta x_i$, approaches 0.

Definition: If f is a function defined on [a,b] the **definite integral** of f from a to b is the number

provided that this limit exists. If it does we say that f is integrable on [a,b].

Theorem: If f is continuous on [a,b] then f in integrable on [a,b].

$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$ is not very easy to calculate!

The good news is that if f is integrable on [a,b] then every Riemann sum must approach the integral no matter how the partitions or sample points are chosen.

This means if a function is integrable (like in the case where it is continuous) we can calculate the integral using any partition and sample points we want, so let's use the easiest one!

We will use the regular partition and right endpoints:

Theorem: If f is integrable on [a,b] then

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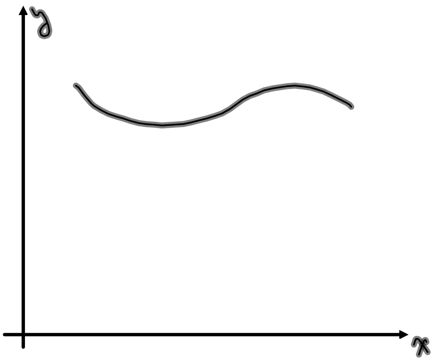
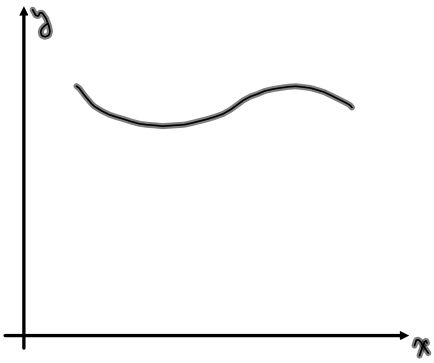
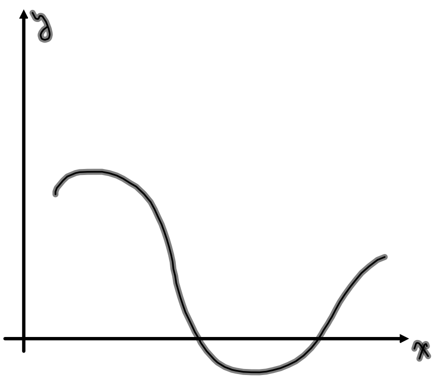
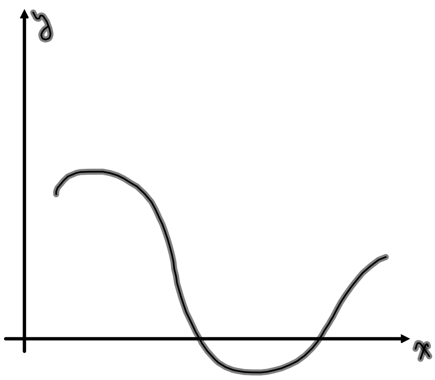
Notice that if f happens to be positive the definite integral $\int_a^b f(x)\mathrm{d}x$ can be interpreted as the area under the curve $y = f(x)$ from a to b.





If f takes on both positive and negative values then the definite integral can be interpreted as a net area, or in other words a difference of areas:





Example:

a) Evaluate the Riemann sum for $f(x) = 2x^3 - x$ taking sample points to be right endpoints and a=0, b=4, n=6

b) Evaluate $\int_0^3 (7x - 2x^3)\,\mathrm{d}x$

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Example: Evaluate $\int_1^5 (x - 4x^2)\,\mathrm{d}x$

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Example: Evaluate $\int_{-2}^{0}(3x^2+2x)\mathrm{d}x$

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Example: Evaluate each of the integrals by interpreting each in terms of areas:

a) $\int_{-1}^{0} \sqrt{1-x^2}\,\mathrm{d}x$

b) $\int_{0}^{5} (2x-4)\,\mathrm{d}x$

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The Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+\dots+f(\bar{x}_n)\right]$$

WHERE $\Delta x = \frac{b-a}{n}$ AND $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$ = MIDPOINT OF $[x_{i-1}, x_i]$.

Example: Use the midpoint rule with n=5 to approximate

$$\int_2^4 \frac{1}{x^2}\,dx$$

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Properties of the Definite Integral

Example: Evaluate $\int_0^1 (7-6x^2)\,\mathrm{d}x$

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Comparison Properties of the Integral

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Example: Use property 8 to estimate $\int_0^1 e^{-x^3}\,\mathrm{d}x$

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