5.4 The Fundamental Theorem of Calculus (Part 1)

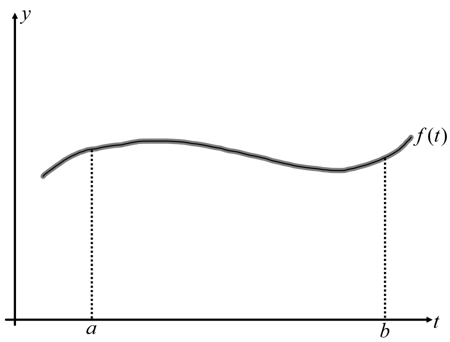
The fundamental theorem of calculus connects the two branches of calculus: differential calculus and integral calculus.

The fundamental theorem deals with functions defined by an equation of the following form (that we saw last time)

$$g(x) = \int_a^x f(x)\mathrm{d}x$$

where f is continuous on [a,b] and x varies between a and b.

Last time we called it A(x) but it's the same function. It gives us the net area from a to b.



Example: If $g(x) = \int_a^x f(t)\mathrm{d}t$ where $a = 1$ and $f(t) = t^2$.

a) Find $g(2)$.

b) Find a formula for $g(x)$.

c) Calculate $g'(x)$.

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The fact that $g'(x) = f(x)$ shouldn't surprise us, we saw last time that $g(x)$ is an antiderivative of $f(x)$.

This is exactly what the FTC part 1 says.

<u>The Fundamental Theorem of Calculus Part 1</u>

If f is continuous on $[a,b]$ then the function $g(x)$ defined by

$$g(x) = \int_a^b f(t)\mathrm{dt} \qquad a \le x \le b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a \le x \le b$.

Notice that another way to write the FTC part 1 is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(t)\,\mathrm{d}t\right] = f(x)$$

Examples: Evaluate the following

1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\int_1^x \left(\dfrac{t}{1+t^2} + \cot^2 t\right)\mathrm{d}t\right]$

2) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\pi}^{x} \arctan t \; \mathrm{d}t\right]$

3) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\pi/2}^{x^2} \sin^3 t \; \mathrm{d}t\right]$

4) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{\cos x}^{x^3+2x} e^{-t^2} \; \mathrm{d}t\right]$

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5) $\dfrac{d}{dx}\left[\int_{\sin(3x^4+2)}^{5} \sqrt{2t-1}\, dt\right]$

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Average Value of a Function

It's easy to take the average of finitely many numbers $y_1, y_2, y_3, \ldots, y_n$

$$y_{\text{ave}} = \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n}$$

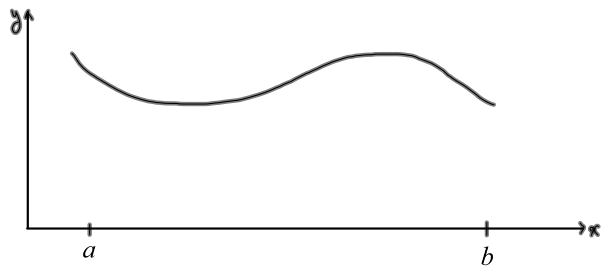
But what if the thing we want to take the average of is continuously changing?

For example, suppose we are driving in a car and we want to find the average velocity over a 30-second period.

And so

If $f(x)$ is the velocity function, what we did was divide the time interval $[a,b]$ into n equal subintervals of length

and we took periodic velocity readings $f(x_1), f(x_2), \ldots, f(x_n)$ where $x_i = a + i\Delta x$. Graphically this looks like:



And so we calculate:

We define the **average value of f** on the interval [a,b] to be

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Example: Find the average value of the function $f(x) = 3x^2 - 2$ on the interval [-2,3].

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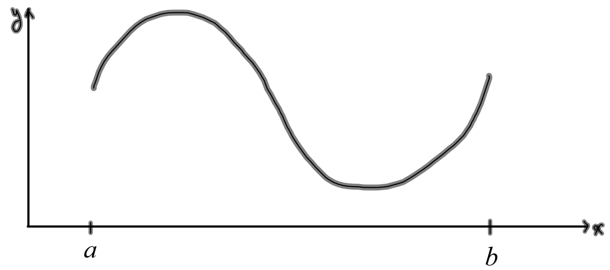
Let's go back to our velocity function:



<u>The Mean Value Theorem for integrals.</u>

If f is continuous on $[a,b]$ then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Notice that the theorem can also be written as

$$\int_a^b f(x)\mathrm{d}x = f(c)(b-a)$$

Example: Let's examine $f(x) = 1 + x^2$ on the interval $[-1,2]$.