Trigonometric Substitutions

In this section we will examine a technique for evaluating integrals containing expressions of the following type

$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$

Previously when making a substitution we would introduce a new variable by letting $u = f(x)$, a function of x. However, for these types of integrals we will want to make a substitution where $x = g(\theta)$. This works because of the substitution rule

$$\int f(x)dx = \int f(g(\theta))g'(\theta)d\theta$$

This kind of substitution is called an inverse substitution. To make our calculations easier we assume that g has an inverse function (that is, g is one-to-one.)

The following is a list of trigonometric substitutions that are useful for the given expression because of the identity on the right. The restriction is to ensure that the function is one-to-one.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = \sec\theta, \ 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

There are two main reasons for the restrictions on the substitutions. For example, let's say we have the expression $\sqrt{a^2 - x^2}$. According to the chart we want to make the substitution $x = a\sin\theta$. Notice that x can take any value between a and -a. We want $a\sin\theta$ to 1) also take those same values and 2) be one-to-one. Since $\sin\theta$ is one-to-one on $-\pi/2 \le \theta \le \pi/2$ but takes on all values between -1 and 1, $a\sin\theta$ takes on all values between -a and a on that interval as required. There are similar arguments for the other two substitutions.

Let's try some examples:

Evaluate the following integrals.

1) $\int \frac{dx}{x^2\sqrt{9 - x^2}}$

2) $\int \frac{dx}{\sqrt{x^2 + 4}}$

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3) $$\int \frac{dx}{(x^2+1)^{3/2}}$$

4) $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2 - 3}}{x} dx$

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5) $\int \frac{1}{x\sqrt{4x^2+3}}dx$

•

6) $\int \frac{x}{\sqrt{x^2-9}}dx$

Completing the square

$x^2 + bx + c$

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7) $\int (x+1)\sqrt{x^2+2x+2}dx$

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8) $\int \frac{\sqrt{1-x^2}}{x^4} dx$

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