Dawson College: Calculus II (SCIENCE): 201-NYB-05-S10: Winter 2016

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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate the definite integral of $f(x) = x^3 + 1$ on $[-1, 2]$ using the definition of the definite integral.

$$x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$\int_{-1}^{2} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(-1+\frac{3i}{n}\right)^3 + 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[-1 + \frac{9i}{n} - \frac{27i^2}{n^2} + \frac{27i^3}{n^3} + 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\sum_{i=1}^{n} \frac{9i}{n} - \sum_{i=1}^{n} \frac{27i^2}{n^2} + \sum_{i=1}^{n} \frac{27i^3}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{n}\sum_{i=1}^{n} i - \frac{27}{n^2}\sum_{i=1}^{n} i^2 + \frac{27}{n^3}\sum_{i=1}^{n} i^3\right]$$

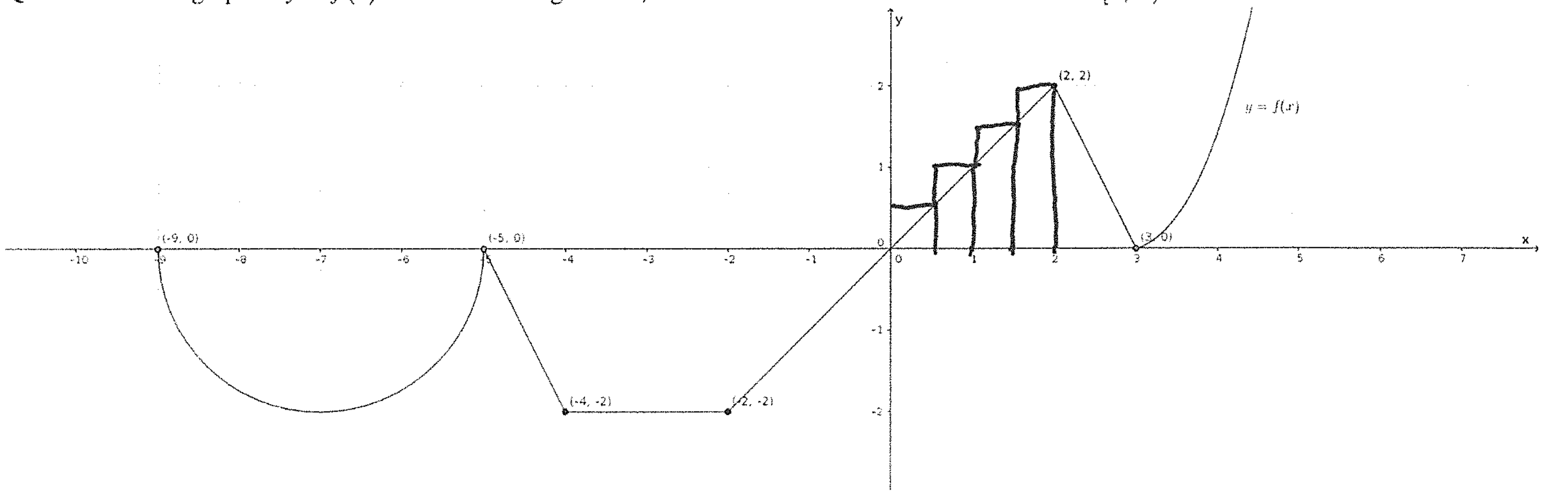
$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{2} - \frac{\overset{9}{\cancel{27}}}{n^{\cancel{2}}} \frac{\cancel{n}(n+1)(2n+1)}{\underset{2}{\cancel{6}}} + \frac{27}{n^{\cancel{3}}} \frac{\cancel{n^2}(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{2}\frac{(n+1)}{n} - \frac{27}{2}\frac{(n+1)}{n}\frac{(2n+1)}{n} + \frac{81}{4}\frac{(n+1)^2}{n^2}\right]$$

$$= \frac{27}{2} - \frac{27}{2}\cdot 2 + \frac{81}{4}$$

$$= \frac{27}{4}$$

Question 2. The graph of $y = f(x)$ consists of straight lines, one semicircle and a curve on the interval $[3,\infty)$.



a. *(5 marks)* Find an approximation of the area under $f(x)$ on the interval $[0,2]$, using the right endpoint as sample points and 4 approximating rectangles. Draw the approximating rectangles. Is the approximation an overestimate or underestimate? Justify.

b. *(5 marks)* Evaluate $\int_{-9}^{3} f(x)\,dx$

c. *(5 marks)* If $\int_{-9}^{4} -3f(x)+2x+1\,dx = 6\pi - \frac{83}{2}$ then determine $\int_3^4 f(x)\,dx$.

a) $\Delta x = \frac{2-0}{4} = \frac{1}{2}$, $x_i = 0 + i\Delta x = \frac{i}{2}$, $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$

Area of $f(x)$ on $[0,2] \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$= \frac{1}{2}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right) = \frac{5}{2}$

The approximation is an overestimate since $f(x)$ is increasing on $[0,2]$ and the function evaluated at the right endpoint is larger than the function evaluated at points left of the right endpoint.

b) $\int_{-9}^{3} f(x)dx = \int_{-9}^{-5} f(x)dx + \int_{-5}^{0} f(x)dx + \int_{0}^{3} f(x)dx$ = − area of half a circle or radius 2 − area of trapezoid + area of triangle

$= -\frac{\pi 2^2}{2} - \frac{2}{2}(2+5) + \frac{1}{2}2(3)$

$= -4 - 2\pi$

c) $\int_{-9}^{4} -3f(x) + 2x + 1\,dx = 6\pi - \frac{83}{2}$

$\int_{-9}^{4} -3f(x)dx + \int_{-9}^{4} 2x+1\,dx = 6\pi - \frac{83}{2}$

$-3\int_{-9}^{4} f(x)dx + \left[x^2 + x\right]_{-9}^{4} = 6\pi - \frac{83}{2}$

$\boxed{-3\left[\int_{-9}^{3} f(x)dx + \int_{3}^{4} f(x)dx\right] + 4^2 + 4 - \left[(-9)^2 + (-9)\right] = 6\pi - \frac{83}{2}}$

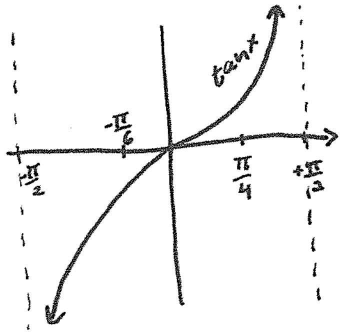
$-3\left[-4 - 2\pi\right] - 3\int_{3}^{4} f(x)dx - 52 = 6\pi - \frac{83}{2}$

$\boxed{-3\int_{3}^{4} f(x)dx = -\frac{3}{2}}$

$\int_{3}^{4} f(x)dx = \frac{1}{2}$

Question 4. *(5 marks)* Evaluate the definite integral:

$$\int_{-\pi/6}^{\pi/4} |\tan(x)|\, dx$$



$$= \int_{-\frac{\pi}{6}}^{0} |\tan x|\, dx + \int_{0}^{\pi/4} |\tan x|\, dx$$

$$= \int_{-\frac{\pi}{6}}^{0} -\tan x\, dx + \int_{0}^{\pi/4} \tan x\, dx$$

$$= \Big[\ln|\cos x|\Big]_{-\frac{\pi}{6}}^{0} + \Big[-\ln|\cos x|\Big]_{0}^{\pi/4}$$

$$= \ln|\cos 0| - \ln\left|\cos \tfrac{-\pi}{6}\right| - \ln\left|\cos \tfrac{\pi}{4}\right| + \ln|\cos 0|$$

$$= -\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{\sqrt{2}} = \ln\frac{2\sqrt{2}}{\sqrt{3}}$$

Question 5. *(5 marks)* Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x}$$

for all $x > 0$.

$$\frac{d}{dx}\left[6 + \int_a^x \frac{f(t)}{t^2}\, dt\right] = \frac{d}{dx}\left[2\sqrt{x}\right]$$

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$$

$$f(x) = x^{3/2}$$

Solve for a:

$$6 + \int_a^x \frac{t^{3/2}}{t^2}\, dt = 2\sqrt{x}$$

$$6 + \int_a^x t^{-1/2}\, dt = 2\sqrt{x}$$

$$6 + \Big[2\sqrt{t}\Big]_a^x = 2\sqrt{x}$$

$$6 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x}$$

$$6 = 2\sqrt{a}$$

$$3 = \sqrt{a}$$

$$9 = a$$

Question 6. Given

$$f(x) = \frac{1}{1+x^2}, \quad [0, \sqrt{3}]$$

a. *(2 marks)* Find the average value of f on the given interval.

b. *(2 marks)* Find c such that $f_{ave} = f(c)$.

c. *(2 marks)* Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

a) $$f_{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\Big[\arctan x\Big]_0^{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\Big[\arctan\sqrt{3} - \arctan 0\Big] = \frac{\pi}{3\sqrt{3}}$$

b) $$f_{ave} = f(c)$$

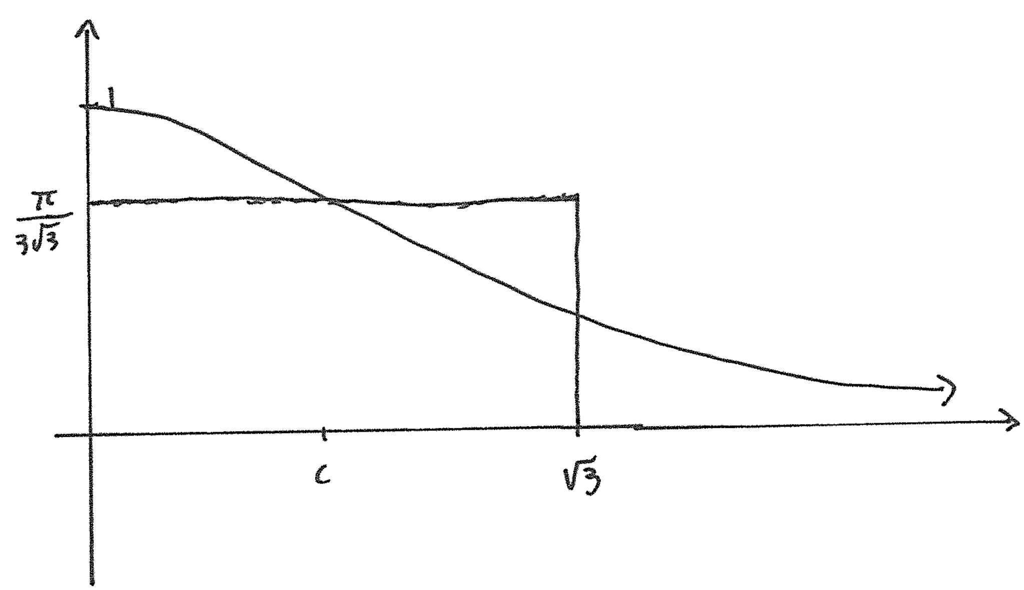
$$\frac{\pi}{3\sqrt{3}} = \frac{1}{1+c^2}$$

$$1+c^2 = \frac{3\sqrt{3}}{\pi}$$

$$c^2 = \frac{3\sqrt{3}}{\pi} - 1$$

$$c = \sqrt{\frac{3\sqrt{3}}{\pi} - 1} \doteq 0.81$$

c)



Question 7.

a. *(4 marks)* Prove: If f is an integrable function on $[a,b]$, then

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

b. *(1 mark)* Prove: If f is an integrable function on $[a,b]$, then

$$\int_a^a f(x)\,dx = 0$$

$$c\int_a^b f(x)dx$$

$$= c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i)\Delta x$$

$$= \int_a^b cf(x)dx$$

$$\int_a^a f(x)\,dx$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } \Delta x = \frac{a-a}{n} = 0$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot 0$$

$$= \lim_{n\to 0}\sum_{i=1}^{n} 0$$

$$= \lim_{n\to 0} 0$$

$$= 0$$

Question 8. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{(1+\sec(x))^2}{\sec(x)}\,dx = \int \frac{1+2\sec x+\sec^2 x}{\sec x}\,dx = \int \frac{1}{\sec x} + 2\frac{\sec x}{\sec x} + \frac{\sec^2 x}{\sec x}\,dx$$

$$= \int \cos x + 2 + \sec x\,dx$$

$$= \sin x + 2x + \ln|\sec x + \tan x| + C$$

Bonus Question. *(5 marks)*
Given the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Show that the solution to the IVP

$$y' = 2xy + \frac{2}{\sqrt{\pi}}, \quad y(0) = 0$$

is the function

$$y = e^{x^2} erf(x).$$

Does the function satisfy the initial condition? $y(0) = e^{0^2} erf(0)$

$$= 1 \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2} dt$$

$$= \frac{2}{\sqrt{2}} \cdot 0$$

$$= 0 \checkmark$$

Does the function satisfy the differential equation?

$$y' = e^{x^2}(2x)\, erf(x) + e^{x^2} \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$= 2x\underbrace{e^{x^2} erf(x)}_{y} + \frac{2}{\sqrt{\pi}}$$

$$= 2xy + \frac{2}{\sqrt{\pi}} \checkmark$$