Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S10: Winter 2016

Name: Y. Lamontagne

Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let f be a continuous odd function and $g(x) = \int_0^x f(t)\,dt$.

a. *(3 marks)* Show that $g(x)$ is an even function.

b. *(2 marks)* Show that

$$\int_{-a}^{a} g(x)f(x)\,dx = 0.$$

a) Since $f(x)$ is odd, $f(-x) = -f(x)$
$-f(-x) = f(x)$

$$g(-x) = \int_0^{-x} f(t)\,dt$$

$u = -t$
$du = -dt$
$-du = dt$
$u(-x) = -(-x) = x$
$u(0) = -0 = 0$

$$= \int_0^{-x} -f(-t)\,dt$$

$$= \int_0^{x} -f(u)(-du)$$

$$= \int_0^{x} f(u)\,du = g(x)$$

$\therefore$ $g(x)$ is even.

b) Let $h(x) = g(x)f(x)$ and lets show that $h(x)$ is odd

$h(-x) = g(-x)f(-x)$
$= g(x)(-f(x))$ since $g(x)$ is even and $f(x)$ is odd.
$= -g(x)f(x)$
$= -h(x)$

$\therefore$ $\int_{-a}^{a} g(x)f(x)\,dx = 0.$

Question 2. *(5 marks)* Apply partial fraction decomposition to the following rational function.

$$R(x) = \frac{x^5 + x - 1}{x^3 + 1}$$

Improper rational function lets perform long division.

$$\begin{array}{r} x^2 \\ x^3+1 \enclose{longdiv}{x^5+0x^4+0x^3+0x^2+x-1} \\ -(x^5 + x^2) \\ \hline -x^2+x-1 \end{array}$$

$\therefore$ $R(x) = x^2 + \dfrac{-x^2+x-1}{x^3+1}$

Lets factor x^3+1, $x=-1$ is a root of x^3+1 since $(-1)^3+1=0$

Hence $x+1$ is a factor of x^3+1

$$\begin{array}{r} x^2 - x + 1 \\ x+1 \enclose{longdiv}{x^3+0x^2+0x+1} \\ -(x^3+x^2) \\ \hline -x^2+0x+1 \\ -(-x^2-x) \\ \hline x+1 \\ -(x+1) \\ \hline 0 \end{array}$$

$\therefore$ $R(x) = x^2 + \dfrac{-x^2+x-1}{(x+1)(x^2-x+1)} = x^2 + \dfrac{-1\cancel{(x^2-x+1)}}{(x+1)\cancel{(x^2-x+1)}}$

$= x^2 - \dfrac{1}{x+1}$

Question 3. *(5 marks)* Evaluate the integral, if possible.

$$\int_0^{\pi/2} \cot^3 x \csc^3 x \, dx = \lim_{a \to 0^+} \int_a^{\pi/2} \cot^3 x \csc^3 x \, dx$$

$$= \lim_{a \to 0^+} \int_a^{\pi/2} \cot^2 x \csc^2 x \cot x \csc x \, dx$$

$$= \lim_{a \to 0^+} \int_a^{\pi/2} (\csc^2 x - 1) \csc^2 x \cot x \csc x \, dx$$

$u = \csc x$

$du = -\csc x \cot x \, dx$

$-du = \csc x \cot x \, dx$

$u(\frac{\pi}{2}) = \csc \frac{\pi}{2} = 1$

$u(a) = \csc a$

$$= \lim_{a \to 0^+} \int_{\csc a}^{1} (u^2 - 1) u^2 (-du)$$

$$= \lim_{a \to 0^+} \int_{\csc a}^{1} u^2 - u^4 \, du$$

$$= \lim_{a \to 0^+} \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_{\csc a}^{1} = \lim_{a \to 0^+} \left[\frac{\csc^5 a}{5} - \frac{\csc^3 a}{3} \right] + \frac{2}{15}$$

diverges to ∞.

Question 4. *(5 marks)* Evaluate the integral, if possible.

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\arctan x}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[[uv]_1^b - \int_1^b v\,du \right]$$

$$u = \arctan x \qquad du = \frac{1}{1+x^2}\,dx$$

$$v = \frac{-1}{x} \qquad dv = \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[\left[\frac{-\arctan x}{x} \right]_1^b - \int_1^b \frac{-1}{x(x^2+1)}\,dx \right]$$

$$= \lim_{b\to\infty} \left[\cancel{\frac{-\arctan b}{b}}^{\to 0} + \underbrace{\frac{\arctan 1}{1}}_{=\frac{\pi}{4}} + \int_1^b \frac{1}{x} - \frac{x}{x^2+1}\,dx \right]$$

Partial Fractions

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Let $x = 0 \Rightarrow A = 1$

$x = 1 \Rightarrow 1 = 2 + B + C$

$-1 = B + C$ ①

$x = -1 \Rightarrow -1 = B - C$ ② $\Rightarrow$ ① = ②

$B + C = B - C$

$C = 0$

$\Rightarrow B = -1$

$$= \lim_{b\to\infty} \left[0 + \frac{\pi}{4} + \left[\ln|x| - \frac{1}{2}\ln|x^2+1| \right]_1^b \right]$$

$$= \frac{\pi}{4} + \lim_{b\to\infty} \left[\ln b - \frac{1}{2}\ln(b^2+1) - \underbrace{\ln 1}_{0} + \frac{1}{2}\ln|1^2+1| \right]$$

I.f. $\infty - \infty$

$$= \frac{\pi}{4} + \ln\sqrt{2} + \lim_{b\to\infty} \ln\left(\frac{b}{\sqrt{b^2+1}}\right) \to 1$$

$$= \frac{\pi}{4} + \ln\sqrt{2} + \lim_{b\to\infty} \ln\left(\sqrt{\frac{\cancel{b^2}}{\cancel{b^2}+1}}\right)$$

$$= \frac{\pi}{4} + \ln\sqrt{2}$$

Question 5. *(5 marks)* Evaluate the integral, if possible.

$$\int \frac{1}{\sqrt{x^2-4x}}\,dx$$

complete the square $\quad x^2-4x = x^2-4x+4-4$
$\qquad = (x-2)^2-4$

$$= \int \frac{1}{\sqrt{(x-2)^2-4}}\,dx$$

$x-2 = 2\sec\theta$
$dx = 2\sec\theta\tan\theta\,d\theta$
$\theta \in [0,\frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$$= \int \frac{1}{\sqrt{(2\sec\theta)^2-4}}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{4(\sec^2\theta-1)}}$$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{\sqrt{4\tan^2\theta}}$$

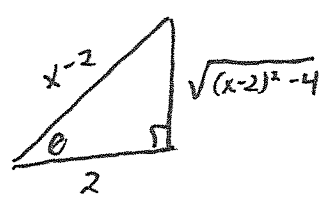
$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2|\tan\theta|}$$

$$= \int \frac{\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{2}\,\cancel{\tan\theta}} \quad \text{since } \theta \in [0,\tfrac{\pi}{2}) \cup [\pi, \tfrac{3\pi}{2})$$

$$= \int \sec\theta\,d\theta$$

$\frac{hyp}{adj} = \frac{x-2}{2} = \sec\theta$

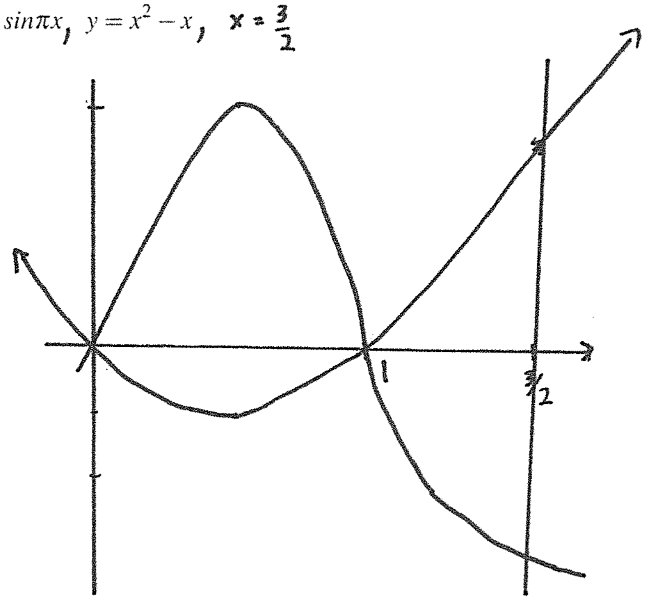
$$= \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\frac{(x-2)}{2} + \frac{\sqrt{(x-2)^2-4}}{2}\right| + C$$

Question 6. *(5 marks)* Sketch the region enclosed by the given curves and find its area. (5)

$y = \sin\pi x, \ y = x^2 - x, \ x = \frac{3}{2}$



$y = x^2 - x$:

x-int: $0 = x^2 - x$

$0 = x(x-1)$

$x = 0 \quad x = 1$

y-int: $y = 0$

vertex: $y = x^2 - x + \frac{1}{4} - \frac{1}{4}$

$= (x - \frac{1}{2})^2 - \frac{1}{4}$

$\therefore$ vertex $(\frac{1}{2}, \frac{-1}{4})$

$$\text{Area} = \int_0^1 \sin\pi x - (x^2 - x)\,dx + \int_1^{3/2} x^2 - x - \sin\pi x\,dx$$

$$= \left[-\frac{\cos\pi x}{\pi} - \frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{\cos\pi x}{\pi} \right]_1^{3/2}$$

$$= \left[-\frac{\cos\pi}{\pi} - \frac{1}{3} + \frac{1}{2} + \frac{\cos 0}{\pi} + \frac{0^3}{3} - \frac{0^2}{2} \right] + \left[\frac{(3/2)^3}{3} - \frac{(3/2)^2}{2} + \frac{\cos\frac{3\pi}{2}}{\pi} - \frac{1}{3} + \frac{1}{2} - \frac{\cos\pi}{\pi} \right]$$

$$= \frac{1}{\pi} - \frac{1}{3} + \frac{1}{2} + \frac{1}{\pi} + \cancel{\frac{9}{8}} - \cancel{\frac{9}{8}} - \frac{1}{3} + \frac{1}{2} + \frac{1}{\pi}$$

$$= \frac{3}{\pi} + \frac{1}{3}$$

Question 7. *(5 marks)* Find the length of the curve

$$y = \int_1^x \sqrt{t^2 - 1}\, dt, \quad 8 \le x \le 64$$

$y' = \sqrt{x^2-1}$ by 2^{nd} FTC

$$S = \int_8^{64} \sqrt{1 + (y')^2}\, dx$$

$$= \int_8^{64} \sqrt{1 + (\sqrt{x^2-1})^2}\, dx$$

$$= \int_8^{64} \sqrt{x^2}\, dx$$

$$= \int_8^{64} |x|\, dx$$

$$= \int_8^{64} x\, dx \quad \text{since } x \in [8, 64]$$

$$= \left[\frac{x^2}{2} \right]_8^{64} = \frac{64^2}{2} - \frac{8^2}{2} = 2016$$

Question 8.[1] *(5 marks)* Evaluate the integral, if possible.

$$\int_0^{\pi/3} \sec x \ln(\sec x + \tan x)\, dx$$

$u = \ln(\sec x + \tan x)$

$du = \dfrac{1}{\sec x + \tan x} \sec x \tan x + \sec^2 x\, dx$

$du = \dfrac{\sec x\ \cancel{(\sec x + \tan x)}}{\cancel{\sec x + \tan x}}\, dx$

$du = \sec x\, dx$

$u(\frac{\pi}{3}) = \ln(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}) = \ln(2+\sqrt{3})$

$u(0) = \ln(\sec 0 + \tan 0) = \ln 1 = 0$

$$= \int_0^{\ln(2+\sqrt{3})} u\, du = \left[\frac{u^2}{2}\right]_0^{\ln(2+\sqrt{3})}$$

$$= \frac{(\ln(2+\sqrt{3}))^2}{2} - \frac{0^2}{2} = \frac{(\ln(2+\sqrt{3}))^2}{2}$$

[1] from a John Abbott final examination

Bonus Question. *(3 marks)*
Integrate.

$$\int \cos\left(\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(i\frac{x}{n}\right)\frac{x}{n}\right)\sin x\,dx$$

$$= \int \cos\left(\int_0^x \sin t\,dt\right)\sin x\,dx$$

$$u = \int_0^x \sin t\,dt$$

$$du = \sin x\,dx$$

$$= \int \cos u\,du$$

$$= \sin u + C$$

$$= \sin\left(\int_0^x \sin t\,dt\right) + C$$