Dawson College: **Calculus II** (Science): 201-NYB-05-S10: Winter 2016

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Test 3

This test is graded out of 41 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

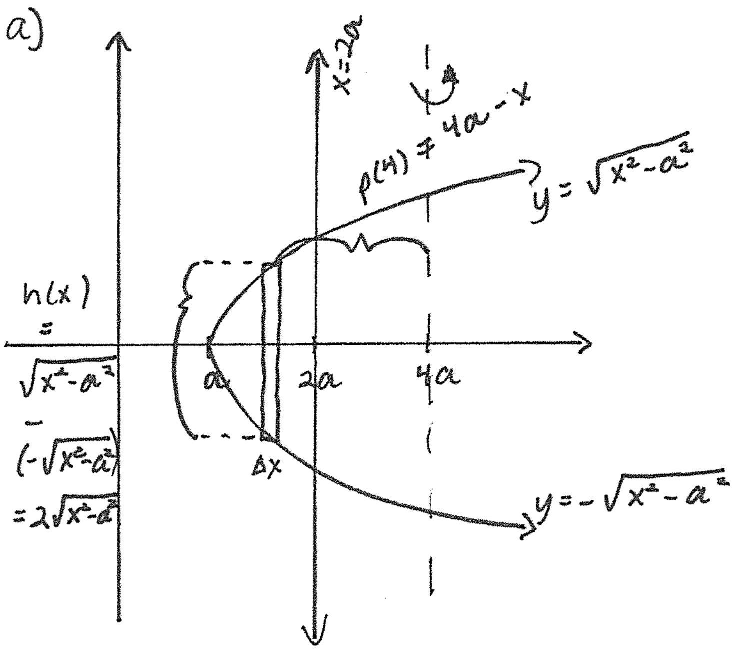
Question 1. For each of the following parts, set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis using the specified method. Sketch the region, draw a representative rectangle, write a representative element and label the sketch completely.

$x^2 - y^2 = a^2$, $x = 2a$; about the $x = 4a$ where $a > 0$.

a. *(5 marks)* Using the shell method.

b. *(5 marks)* Using the washer method.

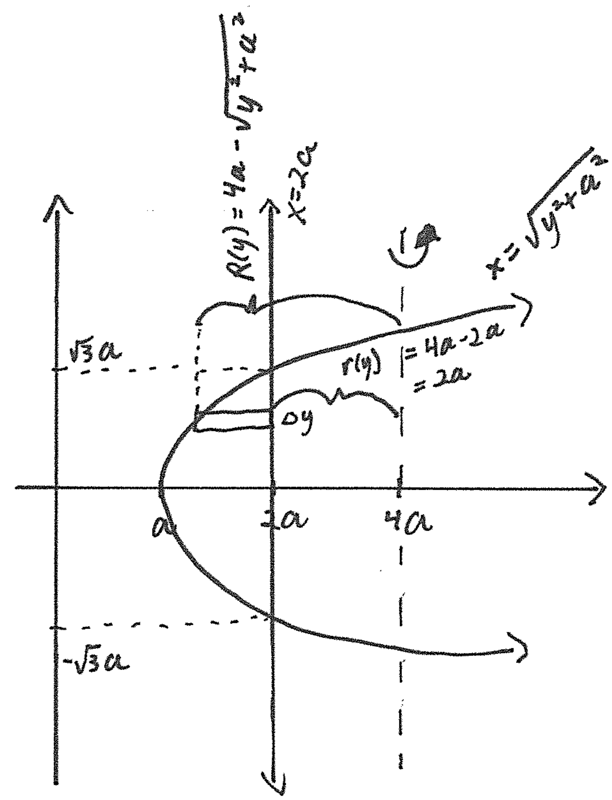
a)



$$\Delta V = 2\pi\, p(x)\, h(x)\, \Delta x$$
$$= 2\pi\,(4a - x)(2\sqrt{x^2-a^2})\,\Delta x$$

$$V = \int_a^{2a} 2\pi(4a-x)(2\sqrt{x^2-a^2})\,dx$$

b)



Sub $x=2a$ into $x^2 - y^2 = a^2$

$$(2a)^2 - y^2 = a^2$$
$$4a^2 - a^2 = y^2$$
$$\pm\sqrt{3}a = y$$

$$\Delta V = \pi\left[(R(y))^2 - (r(y))^2\right]\Delta y$$
$$= \pi\left[(4a - \sqrt{y^2+a^2})^2 - (2a)^2\right]\Delta y$$

$$V = \int_{-\sqrt{3}a}^{\sqrt{3}a} \pi\left[(4a - \sqrt{y^2+a^2})^2 - (2a)^2\right] dy$$

Question 2. Determine whether the statement is true or false. If it is true, explain why. If it is false give an example that disproves the statement.

a. *(2 marks)* If $\{a_n\}$ and $\{b_n\}$ are divergent the $\{a_n + b_n\}$ is divergent.

False,

counter example: $a_n = (-1)^n$, $b_n = (-1)^{n+1}$ are both divergent sequences

but $a_n + b_n = 0$ which converges to 0.

b. *(2 marks)* If $\{a_n\}$ and $\{b_n\}$ are divergent the $\{a_n b_n\}$ is divergent.

False,

Same counter example as 2a)

$a_n b_n = (-1)^n (-1)^{n+1} = -1$ which converges to -1.

c. *(2 marks)* If $\{a_n\}$ is monotonically decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.

Since $a_n > 0$ the sequence is bounded below by 0. Since it is given that a_n is decreasing, it is bounded above by a_1 $\therefore$ a_n is bounded and monotonic

Hence a_n is convergent by the monotonic sequence theorem.

Question 3. *(5 marks)* Determine if each of the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+n+1}{n^2+2n+1}\right)$$

Let $a_n = \ln\left(\frac{2n^2+n+1}{n^2+2n+1}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n^2+n+1}{n^2+2n+1}\right)$$

$$= \ln 2 \neq 0$$

$\therefore$ by the n^{th} term divergence test $\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+n+1}{n^2+2n+1}\right)$

is a divergent series.

Question 4. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} b_n$ is the harmonic series which diverges.

Let $a_n = \tan\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \text{I.f. } \frac{0}{0}$$

$$\overset{\hat{H}}{=} \lim_{n\to\infty} \frac{\sec^2\left(\overset{\to 0}{\cancel{\frac{1}{n}}}\right) \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}}$$

$= \sec^2(0) = 1 > 0$ and finite

$\therefore$ by the limit comparison theorem $\sum_{n=1}^{\infty} a_n$ diverges

since $\sum_{n=1}^{\infty} b_n$ diverges.

Question 5. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2016}^{\infty}(-1)^n\frac{\ln(\ln n)+1}{n\ln n}$$

Lets verify for abs. conv. $\displaystyle\sum_{n=2016}^{\infty}\left|(-1)^n\frac{\ln(\ln n)+1}{n\ln n}\right| = \sum_{n=2016}^{\infty}\frac{\ln(\ln n)+1}{n\ln n}$

Let $f(x)=\dfrac{\ln(\ln x)+1}{x\ln x}$

- $f(x)$ is continuous on $x\geq 2016$
- $f(x)\geq 0$ on $x\geq 2016$

(*) • $$f'(x)=\frac{x\ln x\left(\frac{1}{x\ln x}\right)-[\ln(\ln x)+1](\ln x+\frac{1}{x}x)}{(x\ln x)^2}$$

$$=\frac{1-(\ln(\ln x)+1)(\ln x+1)}{(x\ln x)^2}<0. \text{ for } x\geq 2016$$

∴ $f(x)$ is decreasing for $x\geq 2016$

$$\int_{2016}^{\infty}\frac{\ln(\ln x)+1}{x\ln x}\,dx=\lim_{b\to\infty}\int_{2016}^{b}\frac{\ln(\ln x)+1}{x\ln x}\,dx=\lim_{b\to\infty}\int_{\ln(\ln 2016)+1}^{\ln(\ln b)+1}u\,du$$

$u=\ln(\ln x)+1$

$du=\dfrac{1}{x\ln x}dx$

$u(b)=\ln(\ln b)+1$

$u(2016)=\ln(\ln 2016)+1$

$$=\lim_{b\to\infty}\left[\frac{u^2}{2}\right]_{\ln(\ln 2016)+1}^{\ln(\ln b)+1}$$

$$=\lim_{b\to\infty}\left[\frac{(\ln(\ln b)+1)^2}{2}-\frac{(\ln(\ln 2016)+1)^2}{2}\right]$$ ∞

∴ series diverges since improper integral diverges, by integral test.

Is the series cond. conv $\displaystyle\sum_{n=2016}^{\infty}(-1)^n b_n$ where $b_n=\dfrac{\ln(\ln n)+1}{n\ln n}$

1) $$\lim_{n\to\infty}b_n=\lim_{x\to\infty}\frac{\ln(\ln x)+1}{x\ln x}\overset{\hat{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{x\ln x}}{\ln x+\frac{1}{x}x}=\lim_{x\to\infty}\frac{1}{x\ln x(\ln x+1)}=0$$

2) $b_{n+1}\leq b_n$ by (*)

∴ $\displaystyle\sum_{n=2016}^{\infty}(-1)^n b_n$ conv. by A.S.T.

∴ The series is cond. conv.

Question 6. *(5 marks)* Find the radius and interval of convergence for the power series

$$\sum_{n=10!}^{\infty} \frac{(2x-1)^n}{3^n\sqrt{5n+1}} = \sum_{n=10!}^{\infty} a_n(x)$$

Lets apply the ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty} \left|\frac{\frac{(2x-1)^{n+1}}{3^{n+1}\sqrt{5(n+1)+1}}}{\frac{(2x-1)^n}{3^n\sqrt{5n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{\cancel{n}+1}}{3^{\cancel{n}+1}\sqrt{5n+6}} \cdot \frac{\cancel{3^n}\sqrt{5n+1}}{\cancel{(2x-1)^n}}\right|$$

$$= \lim_{n\to\infty}\left[|2x-1| \frac{\sqrt{5n+1}}{3\sqrt{5n+6}}\right]$$

$$= \frac{|2x-1|}{3} < 1$$

So $\frac{|2x-1|}{3} < 1$ in order to conv

$|2x-1| < 3$

$|x-\frac{1}{2}| < \frac{3}{2} = R$

$\therefore$ the radius of conv. is $\frac{3}{2}$

$|x-\frac{1}{2}| < \frac{3}{2}$

$-\frac{3}{2} < x - \frac{1}{2} < \frac{3}{2}$

$-1 < x < 2$

Lets determine convergence at $x=-1$.

$$\sum_{n=10!}^{\infty} \frac{(2(-1)-1)^n}{3^n\sqrt{5n+1}} = \sum_{n=10!}^{\infty} \frac{(-3)^n}{3^n\sqrt{5n+1}}$$

$$= \sum_{n=10!}^{\infty} \frac{(-1)^n \cancel{3^n}}{\cancel{3^n}\sqrt{5n+1}}$$

$$= \sum (-1)^n b_n$$

① $\lim_{n\to\infty} b_n = 0$

② $b_{n+1} \leq b_n$ since $f(x) = \frac{1}{\sqrt{5x+1}}$

$$f'(x) = \frac{-\frac{1}{2}\cdot 5}{(5x+1)^{3/2}} < 0$$

$\therefore$ by the alternating series test the series converges.

Lets determine convergence at $x=2$.

$$\sum_{n=10!}^{\infty} \frac{(2(2)-1)^n}{3^n\sqrt{5n+1}} = \sum_{n=10!}^{\infty} \frac{\cancel{3^n}}{\cancel{3^n}\sqrt{5n+1}}$$

$$= \sum_{n=10!}^{\infty} \frac{1}{\sqrt{5n+1}} = \sum_{n=10!}^{\infty} c_n$$

$$c_n = \frac{1}{\sqrt{5n+1}} \geq \frac{1}{\sqrt{5n+4n}} = \frac{1}{3\sqrt{n}} = d_n$$

$\sum_{n=10!}^{\infty} d_n$ diverges since it is a p-series where $p=\frac{1}{2}$. $\therefore$ by the comparison test $\sum_{n=10!}^{\infty} c_n$ diverges

$\therefore$ the interval of convergence is $[-1, 2)$

Question 7. *(5 marks)* Find the Maclaurin series for $f(x) = \ln(2+3x)$. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

$$f(x) = \ln(2-3x) \qquad f(0) = \ln 2$$

$$f'(x) = \frac{1}{2+3x}(3) = \frac{3}{2+x} \qquad f'(0) = \frac{3}{2}$$

$$f''(x) = \frac{1}{(2-3x)^2} \cdot -1 \cdot (3)^2 = \frac{-3^2}{(2-3x)^2} \qquad f''(0) = \frac{-3^2}{2^2}$$

$$f'''(x) = \frac{1 \cdot (-1)(-2) \cdot (3)^3}{(2+3x)^3} = \frac{1 \cdot 2 \cdot 3^3}{(2+3x)^3} \qquad f'''(0) = \frac{2!\,3^3}{2^3}$$

$$f^{(4)}(x) = \frac{1 \cdot (-1)(-2)(-3) \cdot (3)^4}{(2+3x)^4} = \frac{-1 \cdot 2 \cdot 3 \cdot (3)^4}{(2+3x)^4} \qquad f^{(4)}(0) = -\frac{3!\,3^4}{2^3}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{1 \cdot (-1) \cdot (-2) \cdot (-3) \cdots (-(n-1))\, 3^n}{(2+3x)^n} = \frac{(-1)(-2)(-3) \cdots (-(n-1)) 3^n}{(2+3x)^n} \qquad f^{(n)}(0) = \frac{(-1)^{n+1}(n-1)!\,3^n}{2^n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = \frac{f^{(0)}(0)\,x^0}{0!} + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!}$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{\frac{(-1)^{n+1}(n-1)!\,3^n}{2^n}\, x^n}{n!}$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{3}{2}\right)^n (n-1)!\, x^n}{n!}$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{3}{2}\right)^n x^n}{n}$$

Bonus Question. *(3 marks)*
Using the $K(\varepsilon)$ definition of the limit to prove that if $a_{2n} \to L$ and $a_{2n+1} \to L$ as $n \to \infty$ then $a_n \to L$ as $n \to \infty$.

For any $\varepsilon > 0$ $\exists K_o(\varepsilon)$ s.t. $\forall n \geq K_o(\varepsilon)$, $|a_{2n+1} - L| < \varepsilon$

since $a_{2n+1} \to L$ as $n \to \infty$.

$\exists K_e(\varepsilon)$ s.t. $\forall n \geq K_e(\varepsilon)$, $|a_{2n} - L| < \varepsilon$

Since $a_{2n} \to L$ as $n \to \infty$.

It follows that if $K(\varepsilon) = \max\{2K_o(\varepsilon), 2K_e(\varepsilon)+1\}$
that $\forall n \geq K(\varepsilon)$, $|a_n - L| < \varepsilon$.