Dawson College: Winter 2022: Linear Algebra (SCIENCE): 201-NYC-05-S7: **Test 1, part 1 of 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. Let A be a square matrix. $A\mathbf{x} = A\mathbf{y}$ only if $\mathbf{x} = \mathbf{y}$, then A __must__ be invertible.

b. A linear system with three equations and two variables __might__ be inconsistent.

c. If A, B are matrices such that $AB = I$ then matrix A __might__ be invertible.

d. If AB is defined, then BA __might__ also defined.

e. If A and B are $n \times n$ matrices such that $AB = B$, then A __might__ be an identity matrix.

f. If A and B are both square matrices such that AB equals BA equals the identity matrix, then B __must__ the inverse matrix of A.

g. If E_1 and E_2 are two elementary matrices, then E_1E_2 __might__ be equal to E_2E_1.

h. For any invertible matrix A, the number of leading ones of the RREF of A __must__ be the same as the number of leading ones of the RREF of A^2.

Question 2.[1] *(5 marks)* Given X, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matrices to be invertible.

$$(3X^{-1}B)^{-1} = C(X + B)$$

$$\frac{1}{3}B^{-1}(X^{-1})^{-1} = CX + CB$$

$$\frac{1}{3}B^{-1}X = CX + CB$$

$$\frac{1}{3}B^{-1}X - CX = CB$$

$$CB = \frac{1}{3}B^{-1}X - CX$$

$$CB = \left(\frac{1}{3}B^{-1} - C\right)X$$

$$\left(\frac{1}{3}B^{-1} - C\right)^{-1}CB = \left(\frac{1}{3}B^{-1} - C\right)^{-1}\left(\frac{1}{3}B^{-1} - C\right)X$$

$$\left(\frac{1}{3}B^{-1} - C\right)^{-1}CB = IX$$

$$\left(\frac{1}{3}B^{-1} - C\right)^{-1}CB = X$$

Question 3.[2] *(5 marks)* Find **all** matrices A, if any, such that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A + A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$

For the two products with A to be defined A has to be a 2×2 matrix

So let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} + \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2a+c & a+2b+d \\ 2c & c+2d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}$$

So
$2a + c = 1$ (1)
$a + 2b + d = 4$ (2)
$2c = 2$ (3) $\Rightarrow c = 1$
$c + 2d = 2$ (4) $\Rightarrow 1 + 2d = 2$
$2d = 1$
$d = \frac{1}{2}$

sub $c = 1$ into (1) $\quad 2a + 1 = 1$
$a = 0$

sub $a = 0, d = \frac{1}{2}$ into (2)
$0 + 2b + \frac{1}{2} = 4$
$2b = \frac{7}{2}$
$b = \frac{7}{4}$

$\therefore A = \begin{bmatrix} 0 & \frac{7}{4} \\ 1 & \frac{1}{2} \end{bmatrix}$

[1] From or modified from a John Abbott final examination or WeBWorK

[2] From a Dawson final examination

Question 4. *(5 marks)* Given

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 2 & -2 & 0 & 3 \\ 3 & 0 & 0 & -2 \\ 5 & -2 & -1 & 0 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 0 & -1 & -1 \\ 3 & 0 & 0 & -2 \\ 5 & -2 & -1 & 0 \end{bmatrix} \sim_{\substack{2R_3 \to R_3 \\ 2R_4 \to R_4}} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 0 & -1 & -1 \\ 6 & 0 & 0 & -4 \\ 10 & -4 & -2 & 0 \end{bmatrix}$$

a. *(5 marks)* Find the reduced row echelon form of the matrix A.

b. *(2 marks)* Suppose that A is the coefficient matrix of a homogeneous linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

c. *(1 mark)* Suppose that A is the coefficient matrix of a homogeneous linear system. Use part c. to find two particular solutions to the system.

d. *(4 marks)* Determine whether or not it is possible to express A as product of elementary matrices times the RREF of A (i.e. $A = F_1 F_2 \ldots F_k R$ where the F_i are elementary matrices and R is the RREF of A). If it is give F_1 and F_k. Justify.

a) continued

$$\sim_{\substack{-3R_1+R_3 \to R_3 \\ -5R_1+R_4 \to R_4}} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 6 & 0 & -13 \\ 0 & 6 & -2 & -15 \end{bmatrix} \sim_{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & -1 & -1 \\ 0 & 6 & -2 & -15 \end{bmatrix} \sim_{-R_2+R_4 \to R_4} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\sim_{\substack{3R_1 \to R_1 \\ -R_3 \to R_3 \\ -2R_3+R_4 \to R_4}} \begin{bmatrix} 6 & -6 & 0 & 9 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim_{\substack{R_2+R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2}} \begin{bmatrix} 6 & 0 & 0 & -4 \\ 0 & 1 & 0 & -13/6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim_{\frac{1}{6}R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & -13/6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d)

$$E_k \cdots E_2 E_1 A = R$$
$$(E_k \cdots E_2 E_1)^{-1}(E_k \cdots E_2 E_1) A = (E_k \cdots E_2 E_1)^{-1} R$$
$$I A = E_1^{-1} \cdots E_k^{-1} R$$
$$A = E_1^{-1} \cdots E_k^{-1} R$$

where $F_1 = E_1^{-1}$ and $F_k = E_k^{-1}$

F_1: $I_4 \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = F_1$

F_k: $I_4 \sim_{6R_1 \to R_1} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = F_k = F_{13}$

b) $[A \mid 0] \sim$ Gauss-Jordan $\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -13/6 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_4 = t$, $t \in \mathbb{R}$

$x_3 = -t$

$x_2 = \frac{13}{6}t$

$x_1 = \frac{2}{3}t$

$\therefore (x_1, x_2, x_3, x_4) = (\frac{2}{3}t, \frac{13}{6}t, -t, t)$, $t \in \mathbb{R}$

c) $t = 0$: $(x_1, x_2, x_3, x_4) = (0,0,0,0)$

$t = 1$: $(x_1, x_2, x_3, x_4) = (\frac{2}{3}, \frac{13}{6}, -1, 1)$

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Question 5.[1] *(5 marks)* Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & k & 6 \\ 0 & 2 & 4k \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3h \\ 2 \end{bmatrix}$. Find the value(s) of h and k, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution,

b. infinitely many solutions,

c. no solution.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & k & 6 & 3h \\ 0 & 2 & 4k & 2 \end{bmatrix} \sim_{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & k & 8 & 3h+1 \\ 0 & 2 & \cancel{4k} & 2 \end{bmatrix} \sim_{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4k & 2 \\ 0 & k & 8 & 3h+1 \end{bmatrix}$$

$$\sim_{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2k & 1 \\ 0 & k & 8 & 3h+1 \end{bmatrix} \sim_{-kR_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2k & 1 \\ 0 & 0 & 8-2k^2 & 3h+1-k \end{bmatrix}$$

a) # var = # leading entries in var. col.

So $8-2k^2 \neq 0$

$4-k^2 \neq 0$

$(2-k)(2+k) \neq 0$

$k \neq 2$ $\quad k \neq -2$

b) # var > # leading entries (and no leading entry in var col)

so $8-2k^2 = 0$ and $3h+1-k = 0$

$3h = k-1$

$h = \frac{k-1}{3}$

$k=2$ $\quad k=-2$

$\therefore$ $k=2$ and $h = \frac{2-1}{3} = \frac{1}{3}$

the other case when $k=-2$ and $h = \frac{-2-1}{3} = -1$

c) Leading entry in constant column

So $8-2k^2 = 0$ and $3h-1-k \neq 0$

$k=2$ $\quad k=-2$

$h \neq \frac{k-1}{3}$

$\therefore$ $k=2$ and $h \neq -1$

the other case when $k=-2$ and $h \neq \frac{1}{3}$

Question 6. A square matrix A is *idempotent* if $A^2 = A$.

a. *(1.5 marks)* Find three idempotent matrices.

b. *(1.5 marks)* Prove that if A is idempotent then A^T is idempotent.

a) $A = 0, I, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

b) premise:

$A^2 = A$

conclusion: $(A^T)^2 = A^T$

LHS $= (A^T)^2$

$= A^T A^T$

$= (AA)^T$

$= (A^2)^T$

$= A^T$ since A is idempotent (premise)

$=$ RHS

Question 7.[3] *(5 marks)* Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

[=>] premise:

A and B are row equivalent

conclusion:

A and B have the same RREF

Since A and B are row equivalent A ~ k elementary row operations ~ B.

Suppose we perform Gauss-Jordan on B ~ l elementary " " ~ R

where R is the RREF of B. It follows that

A ~ k elem. row op. ~ B ~ l elem. row op. ~ R.

∴ A and B have the same RREF.

[<=] premise:

A and B have the same RREF

conclusion:

A and B are row equivalent.

Suppose we perform Gauss-Jordan on A and B

A ~ k elem. row op.* ~ R

B ~ l elem. row op.** ~ R

→ Same R by premise.

So A ~ k elem. row op. (same as *)

~ l elem row op. which are the inverse elem. row op. in reverse order of **

~ B

∴ A and B are row equivalent.

Question 8. *(3 marks)* If A is symmetric and skew-symmetric then $A = 0$.

premise:

A is symmetric $(A^T = A)$

A is skew-symmetric $(A^T = -A)$

conclusion:

From the premises we have

$$A^T = A^T$$
$$A = -A$$
$$2A = 0$$
$$A = 0$$

Bonus Question. *(5 marks)* Enumerate all the solution(s) of $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

given that the numbers are $\mathbb{Z}_3$ instead of $\mathbb{R}$. Operations on the numbers of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

[3] From assigned homework.