Dawson College: Winter 2022: Linear Algebra (SCIENCE): 201-NYC-05-S7: **Test 2, part 1 of 2** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. The **only** calculators allowed are the Sharp EL-531**. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark per blank)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If A is a product of elementary matrices, then $\det(A)$ ___might___ equal one.

b. Let A be a 3×3 matrix, and let B be a 4×4 matrix. If the leading ones of the RREF of A is equal to those of the RREF of B, then $\det(A)$ ___might___ equal zero and det(B) ___must___ equal zero.

c. If $\vec{u}$ and $\vec{v}$ are nonzero vectors and $\text{proj}_{\vec{v}}\vec{u} = \vec{u}$, then $\vec{u}$ ___must___ be parallel to $\vec{v}$.

d. Let $\vec{w}$ be orthogonal to both $\vec{u}$ and $\vec{v}$. Then $\vec{w}$ ___must___ be orthogonal to $\vec{u}+\vec{v}$.

e. The vector $\vec{u}\times(\vec{v}\times\vec{w})$ ___might___ be parallel to the vector $\vec{u}$.

f. The vector $\vec{u}\times(\vec{v}\times\vec{w})$ ___must___ be orthogonal to the vector $2\vec{v}\times(-4\vec{w})$.

Question 2. *(1 mark per blank)* Given A an $n\times n$ matrix and k a non-zero scalar.

a. If A is the reduced row echelon form of a singular matrix then $\det(A) =$ ___0___.

b. If A is an elementary matrix obtained by adding k times one row to an other then $\det(A) =$ ___1___.

c. If A is the identity matrix multiplied by k then $\det(A) =$ ___k^n___.

Question 3. *(5 marks)* [2] Given A, an $n\times n$ matrix such that $\det(A) = 9$ and

$$A^3A^T = 3A^{-1}\text{adj}(A)$$

find n.

$$\det(A^3A^T) = \det(3A^{-1}\text{adj}(A))$$

$$\det(A^3)\det(A^T) = 3^n\det(A^{-1})\det(\text{adj}(A))$$

$$(\det(A))^3\det A = 3^n\frac{1}{\det(A)}(\det(A))^{n-1}$$

$$(\det(A))^5 = 3^n(9)^{n-1}$$

$$9^5 = 3^n(3^2)^{n-1}$$

$$(3^2)^5 = 3^n3^{2(n-1)}$$

$$3^{10} = 3^{n+2n-2}$$

$$10 = n+2n-2$$

$$12 = 3n$$

$$4 = n$$

Question 4. *(5 marks)* Using elementary operations show that

$$-sr\begin{vmatrix} a & b\\ c & d\end{vmatrix} = \begin{vmatrix} sb+2d & rsa+2rc\\ d & rc\end{vmatrix}$$

$$\text{RHS} = \begin{vmatrix} sb+2d & rsa+2rc\\ d & rc\end{vmatrix}$$

$$\overset{-2R_2+R_1\to R_1}{=} \begin{vmatrix} sb & rsa\\ d & rc\end{vmatrix}$$

$$\overset{\frac{1}{s}R_1\to R_1}{=} s\begin{vmatrix} b & ra\\ d & rc\end{vmatrix}$$

$$\overset{\frac{1}{r}C_2\to C_2}{=} sr\begin{vmatrix} b & a\\ d & c\end{vmatrix}$$

$$\overset{C_1\leftrightarrow C_2}{=} -sr\begin{vmatrix} a & b\\ c & d\end{vmatrix}$$

[1] From or modified from a John Abbott final examination

[2] From a Dawson College final examination

Question 4. Given $A^{-1} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & 1 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$.

a. *(5 marks)* Find $\det(A)$.

b. *(2 marks)* Find $\operatorname{adj}(A)$.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\operatorname{adj}(A) = \det(A)\, A^{-1}$$

$$= \frac{1}{-36} \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & 1 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

$$|A^{-1}| = \underbrace{a_{12}C_{12}}_{0} + \underbrace{a_{22}C_{22}}_{0} + a_{32}C_{32} + \underbrace{a_{42}C_{42}}_{0}$$

$$= a_{32}C_{32}$$

$$= -2(-1)^{3+2} \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 5 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{vmatrix} \quad \text{after } R_1 + R_2 \to R_2$$

$$= 2\left[a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}\right]$$

$$= 2(1)(-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = 2(-16) = -36$$

$$\det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{-36}$$

Question 5.[1] *(3 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

If A is a symmetric $n \times n$ matrix where n is even then $\det(A) = 0$.

False,

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric

since $I_2^T = I_2$, $n = 2$ (even)

but $\det(I_2) = 1$.

Question 6. *(3 marks)* Solve only for x_3 using Cramer's Rule.

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 4 \\ 5x_2 - 6x_3 = 7 \\ 8x_3 = 9 \end{cases}$$

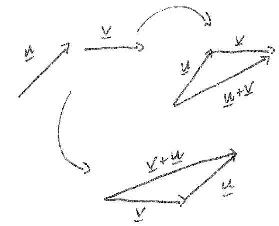
$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -6 \\ 0 & 0 & 8 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}}_{b}$$

$$\det A = 1 \cdot 5 \cdot 8$$

$$\det A_3 = \det \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} = 1 \cdot 5 \cdot 9$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{1 \cdot 5 \cdot 9}{1 \cdot 5 \cdot 8} = \frac{9}{8}$$

Question 7. *(2 marks)* Using sketches illustrate that vector addition is commutative.



Question 8.[1] Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Given: $||\vec{u}|| = 5$, $||\vec{u}+2\vec{v}|| = \sqrt{2}$, $\vec{v}$ and $\vec{u}+3\vec{v}$ are both unit vectors, and the angle between $\vec{u}+2\vec{v}$ and $\vec{u}+3\vec{v}$ is $\pi/4$.

a. *(3 marks)* Find $\vec{u}\cdot\vec{v}$.

b. *(2 marks)* Find $||\vec{u}+\vec{v}||$.

a) $(\underline{u}+2\underline{v})\cdot(\underline{u}+3\underline{v}) = \|\underline{u}+2\underline{v}\|\|\underline{u}+3\underline{v}\|\cos\frac{\pi}{4}$

$\underline{u}\cdot\underline{u} + \underline{u}\cdot(3\underline{v}) + (2\underline{v})\cdot\underline{u} + (2\underline{v})\cdot(3\underline{v}) = \sqrt{2}\cdot 1\cdot\frac{1}{\sqrt{2}}$

$\|\underline{u}\|^2 + 3\underline{u}\cdot\underline{v} + 2\underline{v}\cdot\underline{u} + 6\underline{v}\cdot\underline{v} = 1$

$5^2 + 3\underline{u}\cdot\underline{v} + 2\underline{u}\cdot\underline{v} + 6\|\underline{v}\|^2 = 1$

$5\underline{u}\cdot\underline{v} = 1 - 25 - 6(1^2)$

$\underline{u}\cdot\underline{v} = -6$

b) $\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})$

$= \underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}$

$= \|\underline{u}\|^2 + 2\underline{u}\cdot\underline{v} + \|\underline{v}\|^2$

$= 5^2 + 2(-6) + 1^2$

$= 14$

Question 9. Given two planes: $\mathcal{P}_1:\ x+z=1$ and $\mathcal{P}_2:\ y+z=1$.

a. *(1 mark)* Give a point of intersection of the two planes, by inspection.

$(0,0,1)$ satisfies both P_1 and P_2.

b. *(1 mark)* Give a geometrical argument to explain why the intersection of the two planes is a line.

The normals of both planes are not parallel. Hence the inclination of both planes are different. ∴ the intersection of P_1 and P_2 is a line.

c. *(2 marks)* Find a direction vector for the line of intersection of the two planes without solving for the solution set.

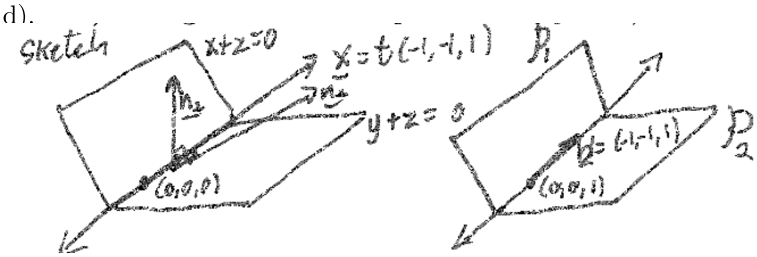
The intersection is parallel to both planes, therefore orthogonal to both normals. $\underline{d} = \underset{\begin{smallmatrix}1\\0\\1\end{smallmatrix}}{\underline{n_1}} \times \underset{\begin{smallmatrix}0\\1\\1\end{smallmatrix}}{\underline{n_2}} = \left(\begin{vmatrix}0&1\\1&1\end{vmatrix}, -\begin{vmatrix}1&1\\0&1\end{vmatrix}, \begin{vmatrix}1&0\\0&1\end{vmatrix}\right) = (-1,-1,1)$

d. *(2 marks)* Find the solution set of the system of linear equations determined by $\mathcal{P}_1$ and $\mathcal{P}_2$ by only using part a) and part c). Justify.

$\underline{x}$ = particular solution of $Ax=b$ + general solution of $Ax=0$

$= (0,0,1) + t(-1,-1,1)$ where $t\in\mathbb{R}$ since the general solution of $Ax=0$ is the line which passes through the origin and $\perp$ $\underline{n_1}, \underline{n_2}$

e. *(1 marks)* Sketch a geometrical interpretation to part d).

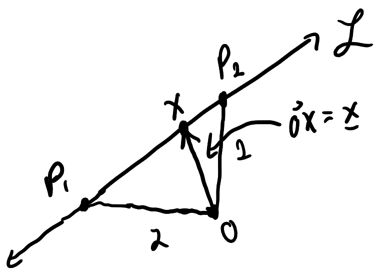


Question 10. Given the points $A(3, 0, -1)$ and $B(3, 1, 0)$.

a. *(1 mark)* Find the equation of the line which passes through A and B.

$\mathcal{L}: \underline{x} = \underline{A} + t\vec{AB} \quad t \in \mathbb{R} \qquad \vec{AB} = \underline{B} - \underline{A} = (3,1,0) - (3,0,-1) = (0,1,1)$

b. *(4 marks)* Find the points on the line which passes through A and B which are $\sqrt{21}$ units away from the origin.



$\mathcal{L}: \underline{x} = (3, 0, -1) + t(0,1,1) = (3, t, -1+t)$

$\|\vec{Ox}\| = \sqrt{21}$

$\|x\| = \sqrt{21}$

$\sqrt{21} = \sqrt{3^2 + t^2 + (-1+t)^2}$

$21 = 9 + t^2 + 1 - 2t + t^2$

$0 = 2t^2 - 2t - 11$

$\therefore$ Two points when $t_1 = \frac{1+\sqrt{23}}{2}$ and $t_2 = \frac{1-\sqrt{23}}{2}$

$P_1 = (3,0,-1) + t_1(0,1,1)$

$P_2 = (3,0,-1) + t_2(0,1,1)$

So $t = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-11)}}{2(2)} = \dfrac{2 \pm \sqrt{4+88}}{4} = \dfrac{2 \pm \sqrt{92}}{4} = \dfrac{2 \pm 2\sqrt{23}}{4} = \dfrac{1 \pm \sqrt{23}}{2}$

Question 11.[3] *(4 marks)* Determine whether the two lines $\mathcal{L}_1 : \vec{x} = (1 - 2t,\ 2 - t,\ 3 + 3t)$ and $\mathcal{L}_2 : \vec{x} = (4 + 3t,\ -1 + t,\ 2 + t)$ intersect, are parallel or are skew lines.

$\mathcal{L}_1: \underline{x} = (1,2,3) + t\underbrace{(-2,-1,3)}_{d_1},\ t \in \mathbb{R}$

$\mathcal{L}_2: \underline{x} = (4,-1,2) + s\underbrace{(3,1,1)}_{d_2},\ s \in \mathbb{R}$

$\mathcal{L}_1$ and $\mathcal{L}_2$ are not $\parallel$ since $d_1 \nparallel d_2$, because they are not multiples of each other.

Do $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect?

$$\left.\begin{aligned} 1-2t &= 4+3s \\ 2-t &= -1+s \\ 3+3t &= 2+s \end{aligned}\right\} \Rightarrow \begin{aligned} -3s-2t &= 3 \\ -s-t &= -3 \\ -s+3t &= -1 \end{aligned}$$

$$\begin{bmatrix} -1 & -1 & -3 \\ -1 & 3 & -1 \\ -3 & -2 & 3 \end{bmatrix}$$

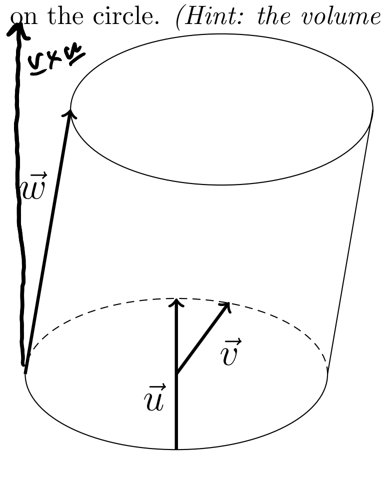
$$\sim \begin{array}{l} -R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \end{array} \begin{bmatrix} -1 & -1 & -3 \\ 0 & 4 & 2 \\ 0 & 1 & 12 \end{bmatrix} \sim -\tfrac{1}{4}R_2 + R_3 \to R_3 \begin{bmatrix} -1 & -1 & -3 \\ 0 & 4 & 2 \\ 0 & 0 & \frac{23}{2} \end{bmatrix}$$

$\therefore$ inconsistent

$\therefore \nexists\ s,t$, for which the lines have the same point. $\therefore \mathcal{L}_1$ and $\mathcal{L}_2$ do not intersect. $\therefore \mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines.

Question 12. *(5 marks)* A cylinder where the sides are not perpendicular to the base (the circle) is called an *oblique cylinder*. Given the oblique cyclinder defined by the given vectors $\vec{u} = (2, 2, 4)$, $\vec{v} = (1, 2, 1)$ and $\vec{w} = (4, 1, 3)$. Find the volume of the oblique cylinder.

Note that from the diagram that $\vec{w}$ is not perpendicular to the base, that $\vec{v}$ is positioned such that its tail is at the center of the circle and its tip lies on the circle, that $\vec{u}$ is positioned such that the vector passes through the center of the circle while its tail and tip lie on the circle. *(Hint: the volume of an oblique cyclinder is equal to the area of the base times the height.)*



area of base $= \pi r^2 = \pi(\|\vec{v}\|)^2 = \pi\left(\sqrt{(1)^2+(2)^2+(1)^2}\right)^2 = 6\pi$

height $= \|\text{proj}_{\vec{v}\times\vec{u}}\ \vec{w}\|$

$\vec{v} \times \vec{u} = \left(\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \right) = (6, -2, -2)$

$= \left\| \dfrac{(4,1,3)\cdot(6,-2,-2)}{(6,-2,-2)\cdot(6,-2,-2)}(6,-2,-2) \right\|$

$= \left\| \frac{16}{44}(6,-2,-2) \right\|$

$= \left\| \frac{4}{11}(6,-2,-2) \right\| = \frac{4}{11}\|(6,-2,-2)\|$

$= \frac{4}{11}\sqrt{36+4+4}$

$= \frac{4}{11}\sqrt{44}$

$= \frac{8}{11}\sqrt{11}$

Volume $=$ base $\times$ height

$= 6\pi \, \frac{8}{11}\sqrt{11} = \frac{48\pi\sqrt{11}}{11}$

Bonus Question. *(5 marks)* If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \le ||\vec{u}||||\vec{v}||$. *Hint: Analyse the squared norm of* $||\vec{u}||\vec{v} - ||\vec{v}||\vec{u}$ *and* $||\vec{u}||\vec{v} + ||\vec{v}||\vec{u}$.

[3]From the assigned homework.