Dawson College: Winter 2022: Linear Algebra (SCIENCE): 201-NYC-05-S7: **Test 3, part 1 of 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If $\vec{v}$ is in Span($\{\vec{u}\}$), then $\vec{u}$ might be in Span($\{\vec{v}\}$).

b. If $A\mathbf{x} = \mathbf{b}$ has two distinct solutions then the columns of A must be linearly dependent

c. The columns of an elementary matrix must form a linearly independent set.

d. If the set $\{\vec{u}, \vec{v}, \vec{w}\}$ spans a plane in $\mathbb{R}^3$, then the set $\{\vec{u}, \vec{v}\}$ might span the same plane.

e. If $\{\vec{u}, \vec{v}\}$ is a basis for a subspace S, then the set $\{6\vec{u} + 3\vec{v}, 10\vec{u} + 5\vec{v}\}$ cannot also be a basis for S.

Question 2.[1] *(1 mark each)*

a. Suppose that $(3, -2, 7)$ and $(-2, a, b)$ is linearly independent then a possibility for (a, b) is $(a, b) =$ $(0,0)$.

b. The vector space of all skew-symmetric $n \times n$ matrices has dimension $\sum_{i=1}^{n} i = \frac{n(n-1)}{2}$.

Question 3.[1] Consider the set $H = \left\{A \mid A \in \mathcal{M}_{2\times 2} \text{ and } \begin{bmatrix}1 & 3\end{bmatrix} A \begin{bmatrix}1 & 3\end{bmatrix}^T = \mathbf{0}\right\}$.

a. *(1 marks)* Find two vectors of H.

b. *(4 marks)* Show that H is a subspace of $\mathcal{M}_{2\times 2}$.

b) Lets apply the subspace test

① Closure under addition.

Let $A, B \in H$ then $A+B \in H$

since $[1\ 3](A+B)[1\ 3]^T$

$= ([1\ 3]A + [1\ 3]B)[1\ 3]^T$

$= [1\ 3]A[1\ 3]^T + [1\ 3]B[1\ 3]^T$

$= 0 + 0$ since $A, B \in H$

$= 0$

② Closure under scalar multiplication

Let $A \in H$ and $k \in \mathbb{R}$ then $kA \in H$

since $[1\ 3](kA)[1\ 3]^T$

$= k\,[1\ 3]A[1\ 3]^T$

$= k\,(0)$ since $A \in H$

$= 0$

$\therefore$ H is a subspace of $M_{2\times 2}$ since it is closed under addition and scalar multiplication.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1 \\ 3\end{bmatrix} = 0$$

$$\begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix} a+3b \\ c+3d \end{bmatrix} = 0$$

$a+3b+3(c+3d) = 0$

$a+3b+3c+9d = 0$

$a = -3b-3c-9d$

then $A = \begin{bmatrix} -3b-3c-9d & b \\ c & d \end{bmatrix} \in H$

a) Let $b=c=d=0$ $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$

Let $b=c=d=1$ $A = \begin{bmatrix} -15 & 1 \\ 1 & 1 \end{bmatrix} \in H$

[1] From or modified from a John Abbott final examination

Question 4.[2] Let $V = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$. And the addition in V is defined by $(a, b) \bigoplus (c, d) = (ad + bc, bd)$ and scalar multiplication in V is defined by $t \bigodot (a, b) = (tab^{t-1}, b^t)$

a. *(1 mark)* $(4, 2) \bigoplus (-5, 1)$ $= (4(1) + 2(-5), 2(1)) = (-6, 2)$

b. *(1 mark)* $-2 \bigodot (1, 2)$ $= (-2(1)(2)^{-2-1}, 2^{-2}) = (\frac{-2}{8}, \frac{1}{4})$

c. *(3 marks)* Demonstrate whether the 4th axiom of vector spaces holds. That is, does the zero vector exists.

Let $\underline{z} = (z_1, z_2)$ and $\underline{u} = (a, b) \in V$

$\underline{u} + \underline{z} = \underline{u}$

$(a, b) + (z_1, z_2) = (a, b)$

$(az_2 + bz_1, bz_2) = (a, b)$

① $az_2 + bz_1 = a$

② $bz_2 = b$

since $b > 0$ $z_2 = 1$

sub into ① $a + bz_1 = a$

$bz_1 = 0$

$z_1 = 0$ since $b > 0$

$\therefore \underline{z} = (0, 1)$ and $\underline{z} \in V$ since $1 > 0$.

Question 5.[3] *(7 marks)* Find a basis for all vectors of the form (a, b, c, d) where $c = a + b$ and $d = a - b$ and state its dimension. Also express $(2, 3, -1, 5)$ relative to the basis found, if possible.

Let $W = \{(a, b, c, d) \mid d = a + b \text{ and } c = a - d\}$

Let $\underline{x} = (a, b, a+b, a-d) \in W$

$= (a, 0, a, a) + (0, b, b, -b)$

$= a\underbrace{(1, 0, 1, 1)}_{\underline{b}_1} + b\underbrace{(0, 1, 1, -1)}_{\underline{b}_2}$

$\therefore \beta = \{\underline{b}_1, \underline{b}_2\}$ span W

And β is linearly independent since $\underline{b}_1$ and $\underline{b}_2$ are not multiples of each other.

$\therefore \beta$ is a basis of W.

$\therefore \dim(W) = 2$

$((2, 3, -1, 5))_\beta$ = undefined

since $(2, 3, -1, 5) \notin W$

because $d \neq a - d$

$5 \neq 2 - 3$

[2]From http://www.math.uwaterloo.ca/ jmckinno/Math225/Week1/Lecture1e.pdf

[3]From the assigned homework.

Question 6.[3] *(4 marks)* Prove: Let $\underbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}_{S}$ be a basis for a vector space V. Show that $\underbrace{\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}}_{S'}$ is also a basis, where $\vec{u}_1 = \vec{v}_1$, $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$, and $\vec{u}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$.

Lets show that span(S) = span(S')

① span(S) ⊆ span(S')

$\underline{v_1} \in$ span(S') since $\underline{v_1} = 1\underline{u_1} + 0\underline{u_2} + 0\underline{u_3}$

$\underline{v_2} \in$ span(S') " $\underline{v_2} = -1\underline{u_1} + 1\underline{u_2} + 0\underline{u_3}$

$\underline{v_3} \in$ span(S') " $\underline{v_3} = 0\underline{u_1} + (-1)\underline{u_2} + 1\underline{u_3}$

$\therefore$ span(S) ⊆ span(S')

② span(S') ⊆ span(S)

$\underline{u_1} \in$ span(S) since $\underline{u_1} = 1\cdot\underline{v_1} + 0\underline{v_2} + 0\underline{v_3}$

$\underline{u_2} \in$ span(S) " $\underline{u_2} = 1\cdot\underline{v_1} + 1\cdot\underline{v_2} + 0\underline{v_3}$

$\underline{u_3} \in$ span(S) " $\underline{u_3} = 1\cdot\underline{v_1} + 1\cdot\underline{v_2} + 1\cdot\underline{v_3}$

$\therefore$ span(S') ⊆ span(S)

$\therefore$ span(S') = span(S)

since the number of vectors in S' = dim(span(S)) and S' spans the span(S) it follows that S' is linearly independent.

$\therefore$ S' is a basis of span(S) = V.

Question 7. *(4 marks)* Determine whether $\left\{ \underbrace{\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 3 & 1 \\ -3 & 5 \end{bmatrix}}_{M_3} \right\}$ is linearly independent.

$0 = c_1M_1 + c_2M_2 + c_3M_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ -3 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & -1 & 0 \\ -3 & 7 & 3 & 0 \\ 5 & 3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \\ -5R_1 + R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 13 & 12 & 0 \\ 0 & -7 & -10 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} 5R_3 \to R_3 \\ 5R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 65 & 60 & 0 \\ 0 & -35 & -50 & \end{bmatrix}$$

$$\sim \begin{array}{l} 13R_2 + R_3 \to R_3 \\ -7R_2 + R_4 \to R_4 \end{array} \begin{matrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & -1 & 0 \end{matrix}$$

$$\sim \begin{array}{l} 3R_4 + R_1 \to R_1 \\ -7R_4 + R_2 \to R_2 \\ -31R_4 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} -\frac{1}{5}R_2 \to R_2 \\ R_3 \leftrightarrow R_4 \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{l} -R_2 + R_1 \to R_1 \\ -R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ only trivial sol.

$\therefore$ set is linearly independent.

Question 8. *(4 marks)* In any vector space V, for any $\vec{u}, \vec{v}, \vec{w} \in V$ prove that if $\vec{v} + \vec{w} = \vec{u} + \vec{w}$ then $\vec{v} = \vec{u}$. *Show every step, justify every step, and cite the axiom(s) used!!!*

$\underline{v} + \underline{w} = \underline{u} + \underline{w}$	
$\underline{v} + \underline{w} + \underline{w}' = \underline{u} + \underline{w} + \underline{w}'$	the additive inverse of $\underline{w}$ exists by axiom 5 (add $\underline{w}'$ on both sides)
$\underline{v} + (\underline{w} + \underline{w}') = \underline{u} + (\underline{w} + \underline{w}')$	associativity, axiom 3
$\underline{v} + \underline{0} = \underline{u} + \underline{0}$	axiom 5
$\underline{v} = \underline{u}$	axiom 4.

Bonus Question. *(1+5 marks)* State and prove the Exchange Lemma.